Topological order and quantum entanglement

Xiao-Liang Qi
*Stanford University*
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Outline

• Introduction to topological order
• Introduction to quantum entanglement
• Quantum entanglement measures of topological order
  1) Topological entanglement entropy.
  2) Entanglement spectrum of some topologically ordered states
  3) Momentum polarization.
  4) Topological uncertainty relation.
Part I: Topological order
Topologically ordered states

- **Topological states of matter**
- **Robust against symmetry breaking?**
  - **NO** → **Symmetry protected topological states (SPT)**
  - **YES** → **Has fractionalization and ground state degeneracy?**
    - **NO** → **“invertible” intrinsic topological states**
    - **YES** → **Topologically ordered states**

Examples: quantum spin Hall, 3D TI
Example 1: Fractional quantum Hall states

- Fractional quantum Hall states are first topologically ordered states discovered in nature.
- To understand fractional quantum Hall states we can start from integer quantum Hall states.
- Hall effect: perpendicular voltage due to Lorentz force.
  \[ j \downarrow x = \sigma \downarrow H E \downarrow y \]
- In strong field and low temperature, we get the quantum Hall effect.
  (von Klitzing ‘80)
• Quantized Hall conductance \( \sigma_{\downarrow H} = ne^2 / h \)

• Reason of the quantization: electron orbits in Lorentz force have quantized energy ---Landau levels. Electrons occupying fully packed Landau levels have a quantized Hall conductance.
• **Edge state picture**

• The quantized Hall conductance is carried by chiral edge states.

• The edge states are “chiral” meaning they only move along one direction.

• **Bulk wavefunction**

• In lowest Landau level, the single electron (in symmetric gauge) has the wavefunction $\psi_{\downarrow n} = z^{\uparrow n} e^{\uparrow - |z|^{\uparrow 2} / 2} l^{\downarrow B^{\uparrow 2}}$

• Many-body wavefunction of the fully occupied Landau level $\prod_{i<j}(z^{\downarrow i} - z^{\downarrow j}) \exp[-1/2 l^{\downarrow B^{\uparrow 2}} \sum i^{\uparrow} |z^{\downarrow i} | ^{\uparrow 2}]$
From integer quantum Hall effect to fractional quantum Hall effect

• Fractional quantum Hall effect (Tsui ‘82) refers to quantization of Hall conductance at fractional values such as $1/3 \frac{e\hbar}{2}$. 

• To understand the physics of fractional quantum Hall state, we can think of the parton picture. (Take the 1/3 state for example)

\[
e = e + e + e
\]

• Each electron is considered as a bound state of 3 partons each with 1/3 charge.
Parton picture and Laughlin state

• Electron density \( n = B / \phi_0 \cdot 1/3 \)

• \[ \frac{1}{3} = \{ \frac{1}{9}, \frac{1}{3}, \frac{1}{3} \} \]

• Parton density \( n_{\downarrow i} = B / \phi_0 \cdot 1/3 , \quad i = 1,2,3 \)

• Parton seems an effective magnetic field \( B/3 \)

• Therefore parton filling \( n_{\downarrow i} / B/3 \phi_0 = 1 \)

• Each parton is in an integer quantum Hall state.
  Hall conductance \( \sigma_{\downarrow Hi} = 1 \cdot (e/3)^2 / h \)

• Total Hall conductance \( \sigma_{\downarrow H} = \sum_{i}\sigma_{\downarrow Hi} = 1/3 e^2 / h \)
Laughlin wavefunction

• Parton wavefunction
\[ \Psi_{\downarrow n} (\{z_{\downarrow i}\}) = \prod_{i<j} (z_{\downarrow i} - z_{\downarrow j}) \exp\left[-\frac{1}{6} l_{\downarrow B}^2 \sum_{i<j} |z_{\downarrow i}|^2 \right], \quad n=1,2,3 \]

• Each parton occupies a Landau level.

• Electron wavefunction
\[ \Psi (\{z_{\downarrow i}\}) = \prod_{n} \Psi_{\downarrow n} (\{z_{\downarrow i}\}) = \prod_{i<j} (z_{\downarrow i} - z_{\downarrow j}) \exp\left[-\frac{1}{2} l_{\downarrow B}^2 \sum_{i<j} |z_{\downarrow i}|^2 \right], \]

  (Laughlin '83)

• Three partons are always bounded into an electron.
Why is the Laughlin state topologically ordered?

• Consider a torus of the fractional quantum Hall state and thread a magnetic flux in the hole.

• Current $j_{\downarrow y} = \sigma_{\downarrow H} E_{\downarrow x}$

• When $\sigma_{\downarrow H} = ne^2/h$, $I_{\downarrow y} = vd/dt (\Phi/\Phi_0)$

• Threading a flux $hc/e$, the system should return to the same state as flux $0$ (because there is no AB phase)

• The charge pumped around the torus is $Q = \nu$

• For $\nu = 1/3$, a fractional charge is pumped through the torus. ➔ One obtains a different ground state.
Fractional excitations

• Three ground states $\Psi \downarrow e (\{z\downarrow i\}) = \Psi \downarrow p \uparrow \phi = 2 n \pi / 3 (\{z\downarrow i\})$

• For the same flux in the torus, there are three different values of flux the parton may see.
Fractional excitations

• This statement about ground state is related to excitations in the system. Cutting the torus open, we obtain a sphere with two punctures.

• Threading a flux $\frac{hc}{e}$ pumps charge $q=1/3$ from bottom puncture to top puncture.

• This is the fractionally charged excitation of this system, named as quasiparticle or quasirole.
Fractional statistics

• The quasiparticle with fractional charge $e/3$ and flux $hc/e$ also has fractional statistics.

• Two particles exchanging position by “braiding” leads to an Aharonov-Bohm phase $\theta = \pi/3$

• Fractional statistics is an intrinsic property of topological order
Example 2: Toric code

- A simple model of topological order (Kitaev 03’, Wen 04’)
- Spin $\frac{1}{2}$ defined on links of a square lattice
  
  \[
  H = -A \prod_+ \uparrow \downarrow \sigma \downarrow z \ ij - B \prod_\square \uparrow \downarrow \sigma \downarrow xij
  \]
- $A>0$, $B>0$
- Ground state satisfies the Gauss law
  \[
  \prod_+ \uparrow \downarrow \sigma \downarrow z \ ij = 1
  \]
- Ground state is a sum over closed loop configurations of $\sigma \downarrow z = -1$.
- The model has topological order.
Topological order of the toric code model

- Topological ground state degeneracy
  \[ H = -A \prod + \uparrow \sigma \downarrow z \ ij \quad -B \prod \uparrow \sigma \downarrow x ij \]

- Sphere

\[ \Rightarrow |G\rangle = \sum_{\text{all Config.}} \]
Topological order of the toric code model

- Torus

Not all configurations can be coupled by the Hamiltonian.

- There are 4 ground states
- Ground states can be labeled by flux in the two directions
  \((0,0), (0,\pi), (\pi,0), (\pi,\pi)\)
Topological order of the toric code model

- Fractionalized excitations
- Charge $e \prod + \uparrow \otimes \sigma \downarrow x = -1$
- Flux $m \prod \square \uparrow \otimes \sigma \downarrow z = -1$
- Braiding statistics $\Theta = \pi$
Toric code model and superconductors

• The toric code model actually is very similar to a two-dimensional superconductor

• *If a 2D superconductor has a finite penetration depth*, it will be equivalent to a toric code model.

• $e \times m$—electron

• $m$—vortex with flux $\frac{hc}{2e}$.

• Actual 2D superconductor has a divergent vortex energy, which is why it’s not strictly a topologically ordered state.
Generic features of topologically ordered states

• From the two examples, we can summarize the generic features of topologically ordered states

• **Topological ground state degeneracy** determined by genus
  --Laughlin state $3 \uparrow g$, Toric code $4 \uparrow g$

• Excitations with fractional statistics

• Fractionalized excitations can be obtained by cutting a torus into a sphere with two punctures. Similar for higher-genus surfaces.
Generic features of topologically ordered states

- **Fusion rule of particles**

- *Two particles together must look like a single particle from far away.*

- \( a \times b = N \downarrow a b \uparrow c c \)

- Laughlin state: \( a \downarrow n \times a \downarrow m = a \downarrow n + m \), \( n,m = 0,1,2 \)

- Toric code: \( e \times e = 1 \), \( m \times m = 1 \),
  \( e \times m = \psi \), \( \psi \times \psi = 1 \)

- \( \psi \) is a bound state of \( e,m \) which is a fermion. (like a superconducting quasiparticle)
Key properties of topologically ordered states

- **Braiding**

  The braiding phase may depend on the fusion channel of $a, b$. In general, it’s denoted as $R_{a \downarrow b \uparrow c}$.

- **Paradox**: With only two particles, what’s the difference between braiding and global rotation?
Topological spin of quasiparticles

• The difference comes from the spin of each particle.
• Braiding phase $= \text{global rotation} - \text{spin of each particle} = \text{spin of the fusion} - \text{spin of each particle}$

\[
R_{ab} R_{ba} = e^{i2\pi \left( h_a + h_b - h_c \right)}
\]

• Braiding is determined by spin of particles.
• Laughlin state $h \downarrow n = n \uparrow 2/6$, toric code $h \downarrow e = h \downarrow m = 0$, $h \downarrow \psi = 1/2$. 
Non-Abelian topologically ordered states

- The two examples we gave are “Abelian” topologically ordered states. The fusion of particles are definite, $a \times b = c$.
- There are non-Abelian states in which particles have multiple fusion channels.
- In non-Abelian states, there is a large Hilbert space for given number of particles.
- The dimension of $N$ particles $a$ is $\approx d \downarrow a \uparrow N$, $d \downarrow a$ is called the quantum dimension of $a$.
- Simplest example:
  - Majorana zero modes $\sigma \times \sigma = 1 + \psi$.
  - Two zero modes can fuse into a fermion occupied state or non-occupied state.
- Quantum dimension $d \downarrow \sigma = \sqrt{2}$.
# Summary of key properties of topological order

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<th>interpretation</th>
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<th>Toric code</th>
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<td>Torus ground state degeneracy</td>
<td><img src="image1.png" alt="Diagram" /></td>
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<tr>
<td>Quasiparticle fusion rule</td>
<td>( a \downarrow n \times a \downarrow m = a \downarrow n + m ), ( n, m = 0, 1, 2 \mod 3 )</td>
<td>e \times m = \psi \quad e \times e = 1 \quad m \times m = 1</td>
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<tr>
<td>Spin of particles</td>
<td><img src="image2.png" alt="Diagram" /></td>
<td>0, 1/6, 2/3</td>
<td>0, 0, 0, ( 1/2 )</td>
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<tr>
<td>Braiding statistics</td>
<td>( R \downarrow n m \uparrow n + m = n m \pi / 3 )</td>
<td>( R \downarrow e m \uparrow \psi = -1 )</td>
<td></td>
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<tr>
<td>Quantum dimension</td>
<td><img src="image3.png" alt="Diagram" /></td>
<td>1, 1, 1</td>
<td>1, 1, 1, 1</td>
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Part II: Entanglement measures of topological order
Overview about quantum entanglement

• General definition: Entanglement is a property of composite quantum system where the joint state cannot be written as a product of states of its component systems. (from www.quantiki.org)

• Simplest example: An EPR pair $|\uparrow\rangle_1|\downarrow\rangle_2 - |\downarrow\rangle_1|\uparrow\rangle_2$

• Topologically ordered states are intrinsically related to quantum entanglement

• Different topological ground states look identical in each part of a torus, but look different on the whole torus. ➔ Topologically ordered ground states must be entangled
Measures of quantum entanglement

- Reduced density matrix
- A state of a system with two partitions
  \[ |\psi\rangle = \sum_{n,m} \psi_{nm} |n\rangle_A \otimes |m\rangle_B, \]
- The average value of an operator \( O_{\downarrow A} \) acting on \( A \) is
  \[ \langle \psi | O_{\downarrow A} | \psi \rangle = \sum_{n,n',m} \psi_{nm} \langle n_{\downarrow A} | O_{\downarrow A} | n \rangle_A \sum_{m'} \psi_{n'm'} m'^* \]
- Reduced density matrix
  \[ \rho_{\downarrow nn'} = \sum_{m} \psi_{nm} \psi_{n'm'} m'^* \]
  determines expectation values of all \( O_{\downarrow A} \)
- In short, \( \rho = tr_{\downarrow B} (|\psi\rangle \langle \psi|) \)
Entanglement entropy and entanglement spectrum

- The von Neumann entanglement entropy
  \[ S = - \text{tr}(\rho \log \rho) \]
- \( S = 0 \) if and only iff \( \rho = |\psi\rangle \langle \psi| \) is a pure state without entanglement.
- For a spin in EPR pair, \( \rho = \frac{1}{2} I \), \( S = \log 2 \)
- Entanglement spectrum (Li&Haldane ’08): eigenvalue spectrum of \( \rho \)
  \[ \text{eig}(\rho) = \{ \lambda \downarrow 1, \lambda \downarrow 2, \ldots, \lambda \downarrow n \} \]
- Entanglement spectrum determines the entanglement entropy
  \[ S = - \sum_{n \uparrow \downarrow} \lambda \downarrow n \log \lambda \downarrow n \] and all other bipartite entanglement properties
- Many more entanglement measures can be defined for more than two partitions
Entanglement measure I: Topological entanglement entropy

• A universal subleading term of the entanglement entropy in a topological state (Levin&Wen ’06, Kitaev&Preskill ‘06)

\[ S_{\downarrow A} = \alpha L_{\downarrow A} - S_{\downarrow \text{topo}} \]

\[ S_{\downarrow \text{topo}} = \log \sqrt{\sum_i d_i \downarrow 2} \]

• Example: Toric code
Topological entanglement entropy

- Entanglement comes from the matching between the configurations in A and its complement.
- Locally, each link crossing the boundary contributes one qubit of entanglement.
- Naively, $S = \sqrt{L} \log 2$
- Actually, not all links are independent, due to the Gauss law $\prod \partial A \uparrow \uparrow \sigma \downarrow z = 1$. Total number of degree of freedom $L \sqrt{A} - 1$
- $S = (L \sqrt{A} - 1) \log 2 \Rightarrow S_{\text{topo}} = \log 2$
Topological entanglement entropy

- In a finite size system it’s difficult to do a fitting and get $S_{\text{topo}}$
- Alternatively, some combinations of entanglement entropies can be used to cancel area law term and obtain $S_{\text{topo}}$
- For example (Kitaev&Preskill ‘06)
  \[ S_{\text{topo}} = S_{\downarrow A} + S_{\downarrow B} + S_{\downarrow C} - S_{\downarrow AB} - S_{\downarrow AC} - S_{\downarrow BC} + S_{\downarrow ABC} \]
- Topological entropy probes a topological order in a single ground state
- In general it does not determine the topological order. More measures are needed.
Entanglement measure 2: entanglement spectrum

- Some topologically ordered states such as fractional quantum Hall state have chiral edge states.
- Similar edge states appear in the entanglement spectrum \( (Li\&Haldane \ ‘08, Qi, Katsura\&Ludwig \ ‘11) \)

\[
\rho_{\downarrow A} \approx e^{\uparrow - \beta H_{\downarrow \text{edge}}} - \text{eig} \left[ \log \right] k
\]
Entanglement measure 2: entanglement spectrum

- Physical reason: gapless edge states are coupled and removed from low energy spectrum.
- As a price to pay, they got entangled and shows up in the entanglement spectrum.
Entanglement measure 3: momentum polarization

- Topological entanglement entropy does not directly probe the topological spin of quasiparticles.
- To probe the spin of a particle, we need to twist a particle.
- Twisting a particle is equivalent to twisting half of a cylinder.
- We want to measure the Berry’s phase obtained in this process.
Momentum polarization

• Consider a lattice model on the cylinder, with lattice translation symmetry $T \downarrow y$ ($T \downarrow y \uparrow L \downarrow y = 1$)

• For a state with quasiparticle $a$ in the cylinder, rotating the cylinder is equivalence to spinning two quasi-particles to opposite directions.

• A Berry’s phase $e^{\pm i2\pi h \downarrow a / L \downarrow y}$ is obtained at the left edge, which is cancelled by an opposite phase at the right.

• Total momentum of the left (right) edge $\pm 2\pi h \downarrow a / L \downarrow y$ ➔ Momentum polarization $P \downarrow M = 2\pi h \downarrow a / L \downarrow y$
Momentum polarization

• Viewing the cylinder as a 1D system, the translation symmetry is an internal symmetry of 1D system, of which the edge states carry a projective representation.

• Ideally we want to measure

$$\tau_y$$

• Difficult to implement. Instead, define discrete translation $$T \downarrow y \uparrow L$$. Translation of the left half cylinder by one lattice constant
Momentum polarization

- Naive expectation: \( T\downarrow y\uparrow L \vert G\downarrow a \rangle \sim e^{i2\pi/L\downarrow y} h\downarrow a \vert G\downarrow a \rangle \)
  contributed by the left edge. However the mismatch in the middle leads to excitations and makes the result nonuniversal.

- **Actual result:**
  \[
  \langle G\downarrow a \vert T\downarrow y\uparrow L \vert G\downarrow a \rangle = \exp[i2\pi/L\downarrow y (h\downarrow a - c/24) - \alpha L\downarrow y]
  \]

- The phase part has a universal subleading term
- \( \alpha \) is independent from topological sector \( a \)
- \( c \): chiral central charge of edge state
- Laughlin state \( c = 1 \)
- Toric code \( c = 0 \)
- Even if we don’t know which sector is trivial \( \vert G\downarrow 1 \rangle \), \( h\downarrow a \) can be determined up to an overall constant by diagonalizing \( \langle G\downarrow n \vert T\downarrow y \vert G\downarrow m \rangle \).
Computation of momentum polarization

• Twist $T^\downarrow y^\uparrow L$ only acts on the left half system

• $\lambda^\downarrow a = \langle G^\downarrow a | T^\downarrow y^\uparrow L | G^\downarrow a \rangle = \text{tr}(\rho^\downarrow L \; T^\downarrow y^\uparrow L)$ is determined by the reduced density matrix of left half cylinder

• Momentum polarization $\lambda^\downarrow a$ is determined by edge states in the entanglement spectrum.

• Analytic calculation of $\lambda^\downarrow a$: Using the fact that the entanglement density matrix $\rho^\downarrow A = \exp[-\beta H^\downarrow edge]$ and $H^\downarrow edge$ is a conformal field theory.

• $\rho^\downarrow L$ describes a cylinder with different temperature on two boundaries.

• Only right boundary has finite “temperature” due to entanglement with the other half.
• **Numerical computation of** $\lambda \downarrow \alpha$

• 1. Kitaev honeycomb model (Kitaev ‘06). Can be calculated by mapping to free fermions coupled to $\mathbb{Z}_2$ gauge field

$$H = -\sum_x \text{link} \downarrow \sigma \downarrow x \sigma \downarrow j \uparrow x - \sum_y \text{link} \downarrow \sigma \downarrow j \uparrow y$$

- $\sum_z \text{link} \downarrow \sigma \downarrow j \downarrow z \sigma \downarrow j \uparrow z$

• Results agree with the expectation $c = 1/2, h \downarrow \sigma = 1/16, h \downarrow \psi = 1/2$
• **Numerical computation of** $\lambda \downarrow \alpha$

• 2. Fractional Chern insulators (FCI, i.e. lattice fractional quantum Hall states)

Similar to Laughlin state, FCI ground states can be constructed by partons $|G\rangle = P|G\downarrow 1 \rangle \otimes |G\downarrow 2 \rangle$

Such wavefunctions can be studied by Monte Carlo.

- $\lambda \downarrow \alpha = |\lambda \downarrow \alpha | e^{i \theta \downarrow \alpha}$

- Fitting

$$\theta \downarrow \alpha \downarrow y = -\text{Im} \alpha \downarrow y \uparrow 2 + 2\pi (h \downarrow \alpha \cdot \downarrow y \uparrow)$$

(Tu & Zhang & Qi, ’12, Zhang & Qi, ‘13)
Momentum polarization in more generic geometries

- By studying the momentum polarization on a cone and varying the cone angle, the central charge $c$ contribution can be determined.
- $c$ contribution is geometrical, while $h\downarrow a$ is topological.
- Verified by momentum polarization of a $c$MPS state for the Pfaffian wavefunction (Mong, Zaletel, Qi)
Entanglement measure 4: Topological uncertainty relation

- Topological sector can be measured by quasiparticle paths around the torus
- For Laughlin state, taking $q = e/3$ particle around the torus measures the flux.
- The measurement can be done at any loop $\Rightarrow$ A long range order of string order parameter.
- Long-range correlation between loop operators $\langle \phi(r_{\downarrow 1})\phi(r_{\downarrow 2}) \rangle = 1$
- Similar to classical order in a ferromagnet $\langle S(r_{\downarrow 1})S(r_{\downarrow 2}) \rangle = M^{\uparrow 2}$

Chao-Ming Jian, Isaac Kim & XLQ, in preparation
Comparison between conventional order and topological order

- Spontaneous symmetry breaking leads to classical long-range order
  \[ H = -J \sum_i \sigma_i^\uparrow \sigma_{i+1}^\uparrow \sigma_i^\downarrow \text{ Ising model} \]
- Ground states \(|\uparrow\uparrow...\uparrow\rangle, |\downarrow\downarrow...\downarrow\rangle\)
- Comparison with topological order

Spin eigenstate \(|\uparrow\uparrow...\uparrow\rangle\) and \(|\downarrow\downarrow...\downarrow\rangle\)

Spin correlation \(\langle \sigma_i^\downarrow \sigma_{i+1}^\downarrow \rangle = 1\)

Flux eigenstate \(|\phi\rangle\), \(\phi = 0, 2\pi/3, 4\pi/3\)

Flux correlation between two loops \(\langle \phi(r_{i1})\phi(r_{i2}) \rangle = 1\)
Comparison between conventional order and topological order

• Topological order is like a conventional order after “dimensional reduction” to lower dimension
• Is that it? What’s the key difference between topological order and classical long-range order?
• A torus can be reduced to 1D in two different ways
• Two kinds of long-range correlations for loops $\phi \downarrow x$ and $\phi \downarrow y$.
• Each looks like a long-range order but they don’t commute.
Non-commuting long-range correlations between loop operators

- Measuring $\phi \downarrow x$ requires to take a quasiparticle going around loop X.
- Quasiparticle carries charge $e/3$ and flux $2\pi/3$ (i.e. $hc/e$)
- $\Rightarrow$ Flux in loop Y is changed by $2\pi/3$.
- $e^{i\Phi \downarrow x} e^{i\Phi \downarrow y} = e^{i\Phi \downarrow y} e^{i\Phi \downarrow x} e^{i2\pi/3}$
- Eigenstates of $\phi \downarrow x$ is superposition of $\phi \downarrow y$ eigenstates
- $|\phi \downarrow x = 0\rangle = 1/\sqrt{3} \sum_{n=0}^{\infty} |2 \gg |\phi \downarrow y = 2\pi\rangle$
Non-commuting long-range correlations between loop operators

A spontaneous symmetry breaking state $|\uparrow\downarrow\rangle$

- Lesson: Topological order can be understood as long-range order of non-commuting loop operators
Topological uncertainty relation

• Non-commuting operators such as \([x,p]=i\) lead to Heisenberg’s uncertainty relation.

• For a topological order, loop operators on the torus cannot be simultaneously diagonalized.

• We can define a quantum entanglement measure using this intuition.

• Define mutual information between two regions on a torus

\[
I_{\downarrow}X_{\downarrow1}X_{\downarrow2} = S_{\downarrow}X_{\downarrow1} + S_{\downarrow}X_{\downarrow2} -
\]

• Mutual information measures correlation between the two regions.
Mutual information measures long-range order

- An EPR pair $|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle$, $S\downarrow _1 = S\downarrow _2 = \log_2$, $S\downarrow _{12} = 0$, $I\downarrow _{12} = 2\log_2$ is maximal.
- For classical long-range order

\[ I\downarrow _{12} = 0 \]

\[ I\downarrow _{12} = 0 \]

\[ I\downarrow _{12} = \log_2 \]
Topological uncertainty relation

• For topological order, a state $|\phi \downarrow x = 0\rangle$ has

\[ I \downarrow X = I \downarrow X \downarrow 1 \ X \downarrow 2 = 0, \ I \downarrow Y = I \downarrow Y \downarrow 1 \]

• Alternative, if we take $|\phi \downarrow y = 0\rangle$, it has

\[ I \downarrow X > 0, \ I \downarrow Y = 0. \]

• $I \downarrow X$ and $I \downarrow Y$ cannot simultaneously vanish.

• $I \downarrow X + I \downarrow Y$ has a lower bound, as a consequence of the uncertainty relation (Jian, Kim & XLQ ‘15)

\[ I \downarrow X + I \downarrow Y \geq -2 \log_{\text{max} \rightarrow n,m} |S \downarrow nm| \]

• $S \downarrow nm = n \downarrow X \ m \downarrow Y$ the transformation matrix between the two basis, also known as the modular $S$-matrix.
Topological uncertainty relation

• For a generic ground state
  
  \[ |\Psi\rangle = \sum_{n \uparrow} |a \downarrow n \rangle \langle n \downarrow X| = \sum_{n \uparrow} |b \downarrow n \rangle \langle n \downarrow Y| , \]

  \[ I_{\downarrow X} = -\sum_{n \uparrow} |a \downarrow n \rangle \langle \uparrow | 2 \log |a \downarrow n \rangle \langle \uparrow | 2 , \]

  \[ I_{\downarrow Y} = -\sum_{n \uparrow} |b \downarrow n \rangle \langle \uparrow | 2 \log |b \downarrow n \rangle \langle \uparrow | 2 . \]

• Allowed values of \((I_{\downarrow X}, I_{\downarrow Y})\) for some examples:

  - Laughlin 1/3
  - SU(2)_2
  - Fibonacci
More general topology

• On a more generic manifold, any two loops $L\downarrow 1$ and $L\downarrow 2$ with nontrivial intersection has non-commuting loop operators

• $\min(I\downarrow 1 + I\downarrow 3) > 0, \min(I\downarrow 2 + I\downarrow 3) > 0, \min(I\downarrow 1 + I\downarrow 2) = 0$
Summary of topological uncertainty relations

• In general, we can compute mutual information between regions in the system.

• Topological order means
  1) Contractible regions have no mutual information.
  2) Non-contractible regions generically have nonzero mutual information.
  3) When two pairs of loops \((X_1, X_2)\) and \((Y_1, Y_2)\) have nonzero intersection, 
     \(\mathcal{I}_{\downarrow X} + \mathcal{I}_{\downarrow Y}\) have finite lower bound.

• This approach provides a general characteristics of topological order which can be generalized to higher dimensions.

• A direct measure of “long-range entanglement”.
Summary

• Topologically ordered states are states of matter with ground state degeneracy, fractional statistics etc.

• Topological order is intrinsically related to quantum entanglement.

• Topological order is difficult to probe, and quantum entanglement provides helpful characteristics.

• Different quantum entanglement measures can be defined to characterize topologically ordered states, such as topological entanglement entropy, entanglement spectrum, momentum polarization and topological uncertainty relation.

• Many more open questions in higher dimensions.
Key references

Roger Mong, Michael Zaletel and Xiao-Liang Qi, in preparation
Chao-Ming Jian, Isaac Kim and Xiao-Liang Qi, in preparation
Thanks!