Topological order a quantum entanglement

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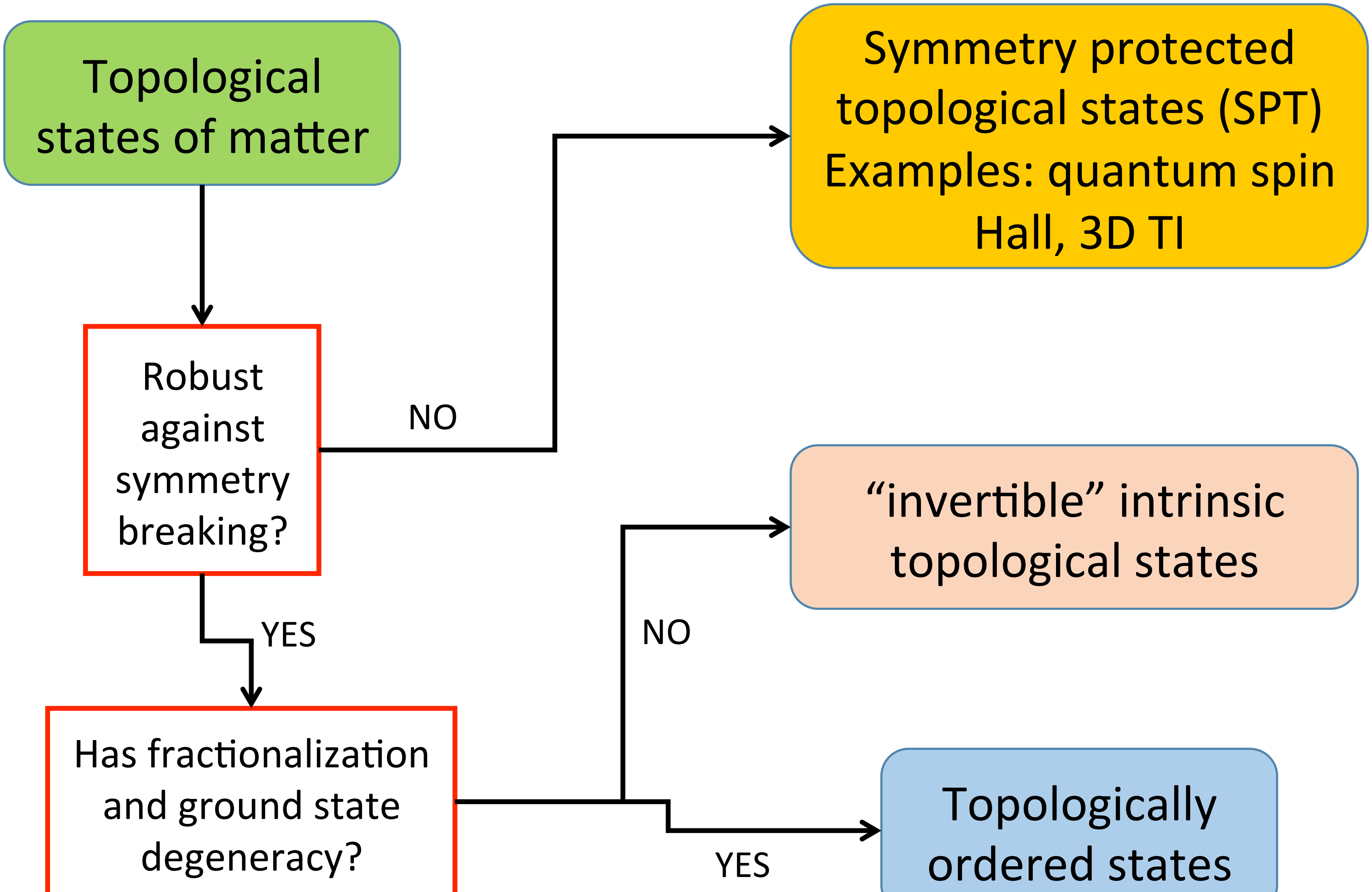
Outline

- Introduction to topological order
- Introduction to quantum entanglement
- Quantum entanglement measures of topological order
  1) Topological entanglement entropy.
  2) Entanglement spectrum of some topologically ordered states
  3) Momentum polarization.
  4) Topological uncertainty relation.

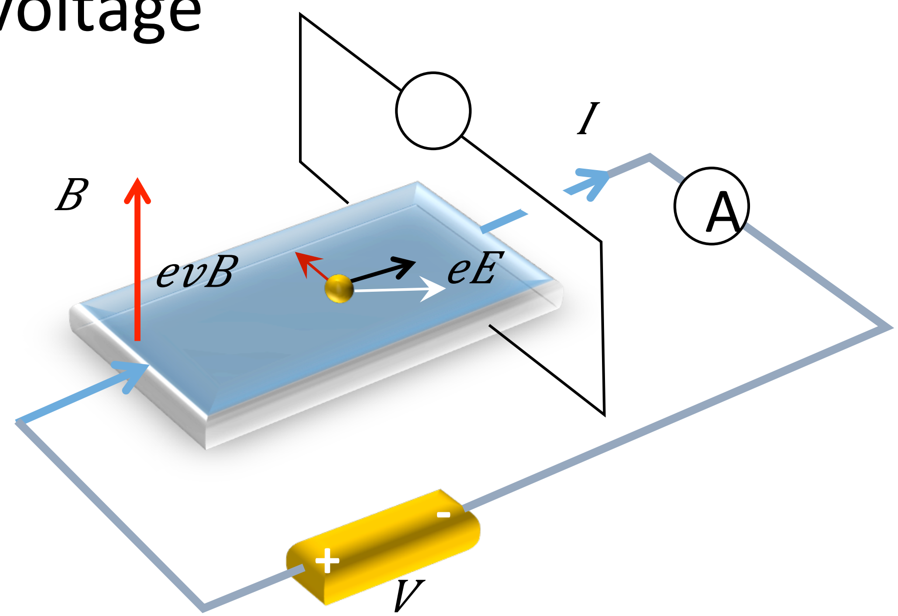
Part I: Topological order

Topologically ordered states

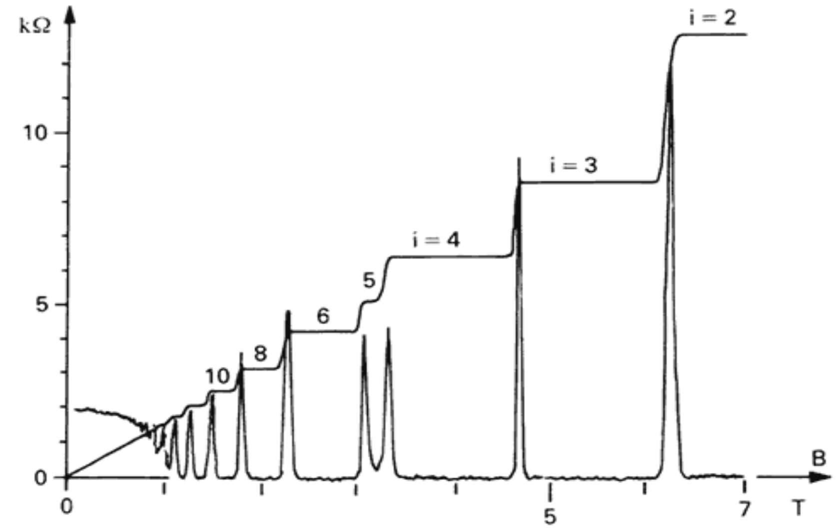


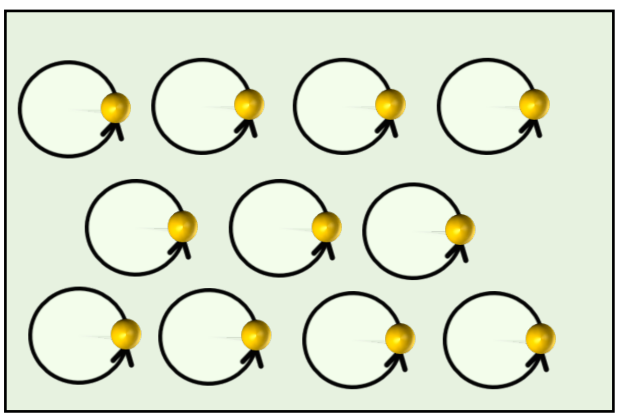
Example 1: Fractional quantum Hall states

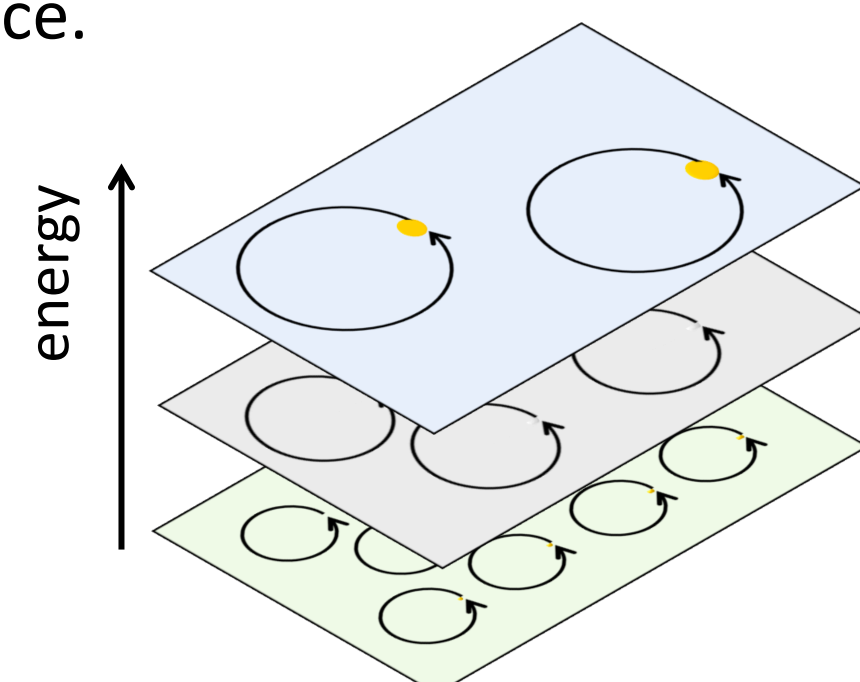
- Fractional quantum Hall states are first topologically ordered states discovered in nature.
- To understand fractional quantum Hall states we can start from integer quantum Hall states.
- Hall effect: perpendicular voltage due to Lorentz force.
  $j_x = \sigma_H E_y$
- In strong field and low temperature, we get the quantum Hall effect.
  (von Klitzing '80)



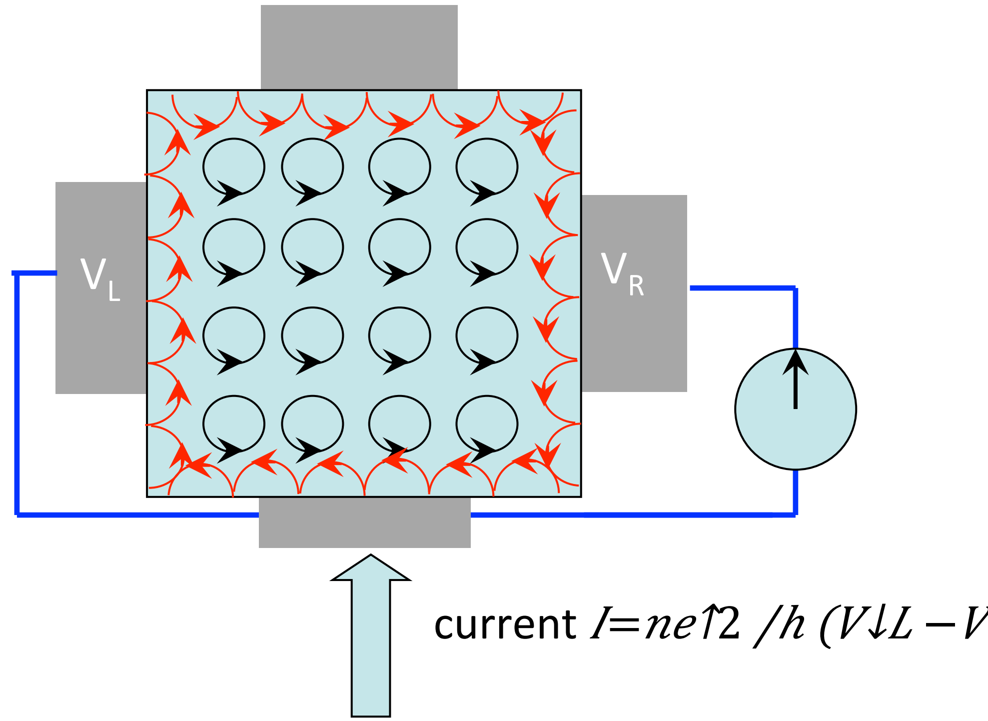
- Quantized Hall conductance $\sigma_H = ne^2/h$
- Reason of the quantization: electron orbits in Lorentz force have quantized energy ---Landau levels. Electrons occupying fully packed Landau levels have a quantized Hall conductance.







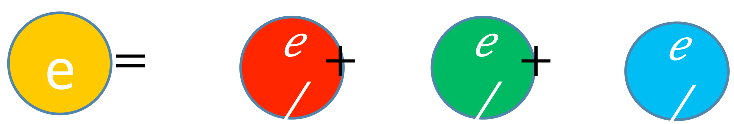
- **Edge state picture**
- The quantized Hall conductance is carried by chiral edge states.
- The edge states are "chiral" meaning they only move along one direction.



- **Bulk wavefunction**
- In lowest Landau level, the single electron (in symmetric gauge) has the wavefunction $\psi_n = z^n e^{-|z|^2/2 l_B^2}$
- Many-body wavefunction of the fully occupied Landau level $\prod_{i<j}(z_i - z_j)\exp\left[-\frac{1}{2l_B^2}\sum_i |z_i|^2\right]$

From integer quantum Hall effect to fractional quantum Hall effect

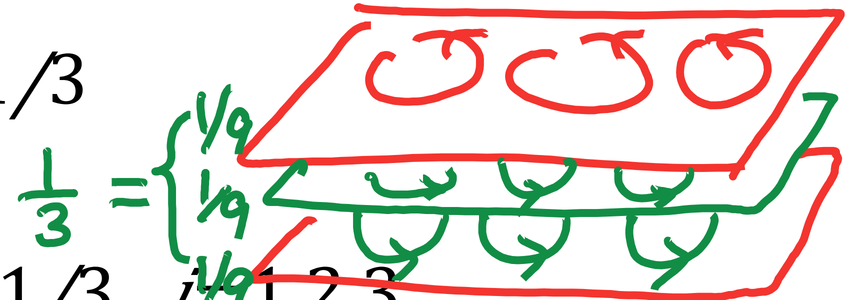
- Fractional quantum Hall effect (Tsui '82) refers to quantization of Hall conductance at fractional values such as $1/3\ e^2/h$.
- To understand the physics of fractional quantum Hall state, we can think of the parton picture. (Take the 1/3 state for example)



- Each electron is considered as a bound state of 3 partons each with 1/3 charge.

Parton picture and Laughlin state

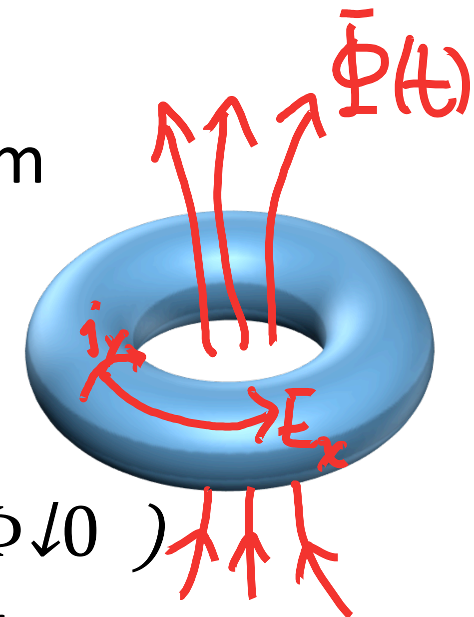
- Electron density $n = \frac{B}{\phi_0} \cdot 1/3$
- ➔
- Parton density $n_i = \frac{B}{\phi_0} \cdot 1/3$, $i = 1,2,3$
- Parton seems an effective magnetic field $B/3$
- Therefore parton filling $n_i / \left(\frac{B}{3\phi_0}\right) = 1$
- Each parton is in an integer quantum Hall state. Hall conductance $\sigma_{Hi} = 1 \cdot (e/3)^2 / h$
- Total Hall conductance $\sigma_H = \sum_i \sigma_{Hi} = 1/3\, e^2 / h$



Laughlin wavefunction

- Parton wavefunction $\Psi_n(\{z_i\}) = \prod_{i<j} (z_i - z_j) \exp\left[-\frac{1}{6 l_B^2} \sum_i |z_i|^2\right]$, n=1,2,3
- Each parton occupies a Landau level.
- Electron wavefunction
- $\Psi(\{z_i\}) = \prod_n \Psi_n(\{z_i\}) = \prod_{i<j} (z_i - z_j)^3 \exp\left[-\frac{1}{2 l_B^2} \sum_i |z_i|^2\right]$,
- (Laughlin '83)
- Three partons are always bounded into an electron.

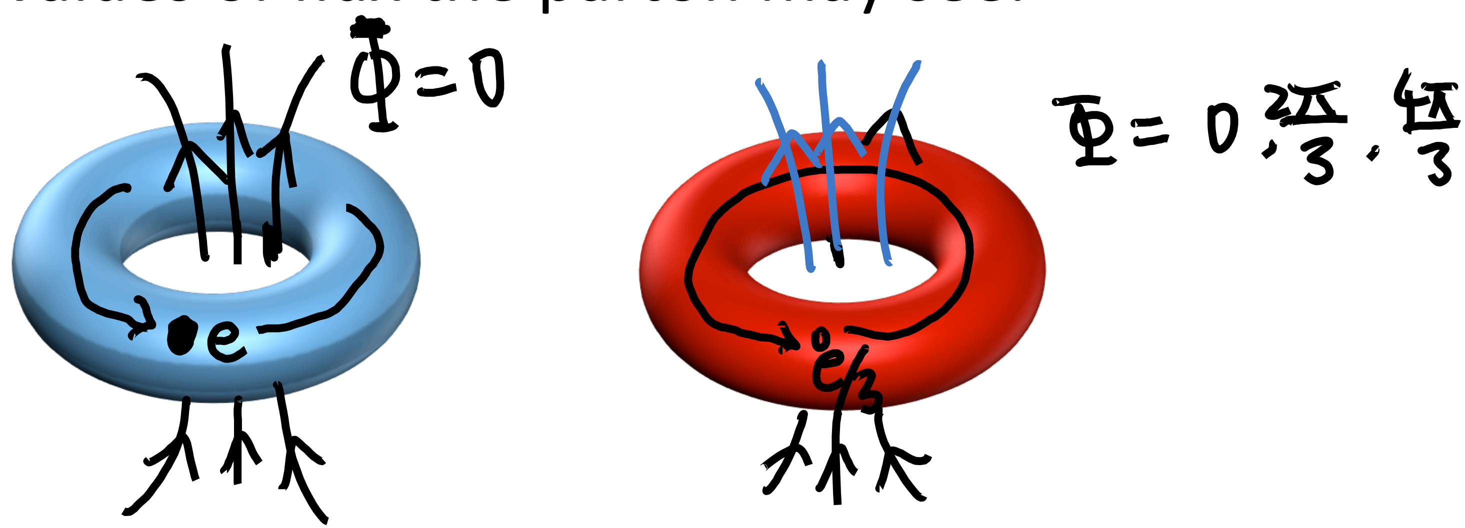
Why is the Laughlin state topologically ordered?



- Consider a torus of the fractional quantum Hall state and thread a magnetic flux in the hole.
- Current $j_y = \sigma_H E_x$
- When $\sigma_H = \nu e^2/h$, $I_y = \nu d/dt\,(\Phi/\Phi_0)$
- Threading a flux hc/e, the system should return to the same state as flux 0 (because there is no AB phase)
- The charge pumped around the torus is $Q = \nu$
- For $\nu = 1/3$, a fractional charge is pumped through the torus. ➔One obtains a different ground state.

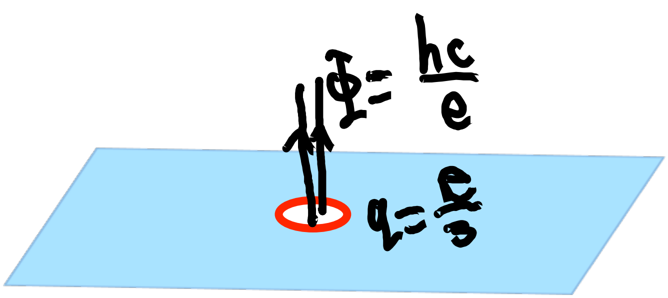
Fractional excitations

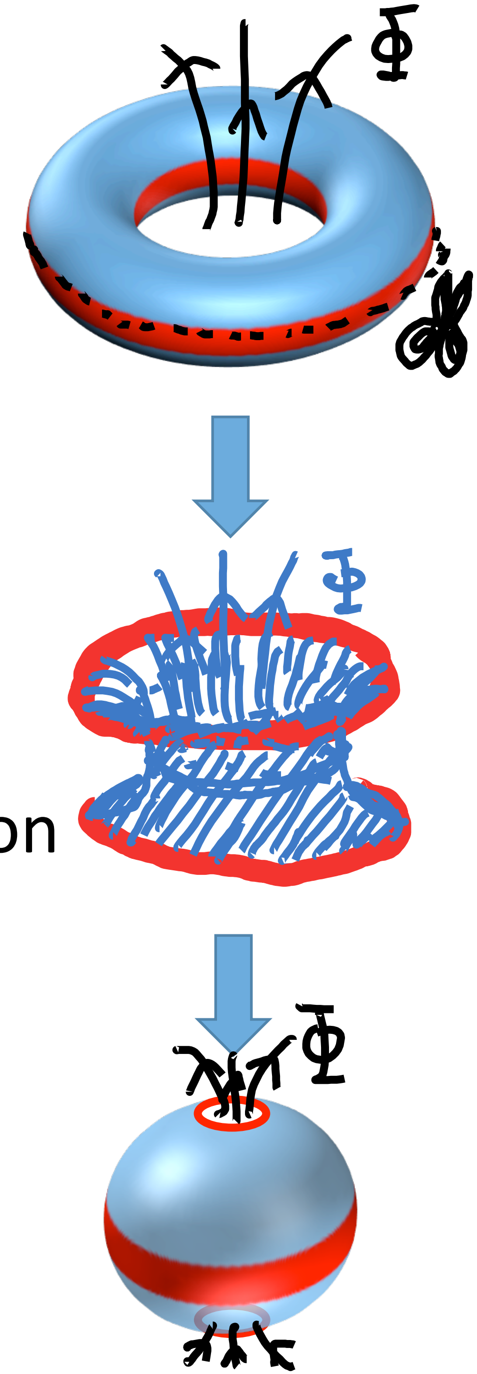
- Three ground states $\Psi_e(\{z_i\}) = \Psi_p^{\phi=2n\pi/3}(\{z_i\})$
- For the same flux in the torus, there are three different values of flux the parton may see.



Fractional excitations

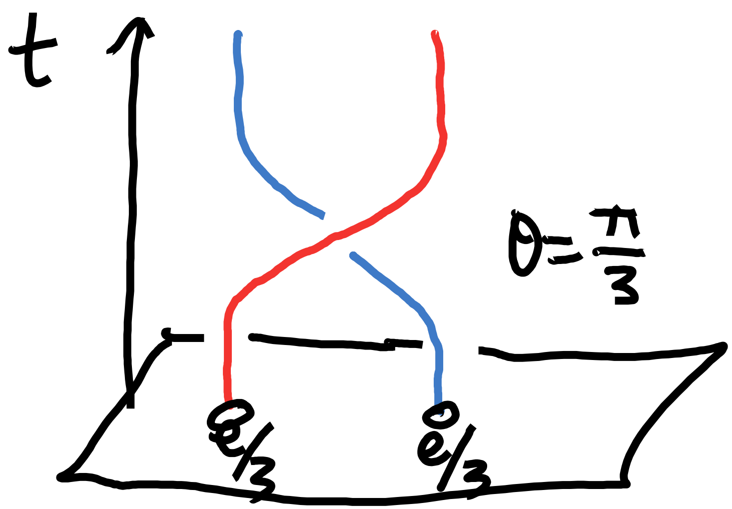
- This statement about ground state is related to excitations in the system. Cutting the torus open, we obtain a a sphere with two punctures
- Threading a flux hc/e pumps charge $q=1/3$ from bottom puncture to top puncture
- This is the fractionally charged excitation of this system, named as quasiparticle or quasihole.



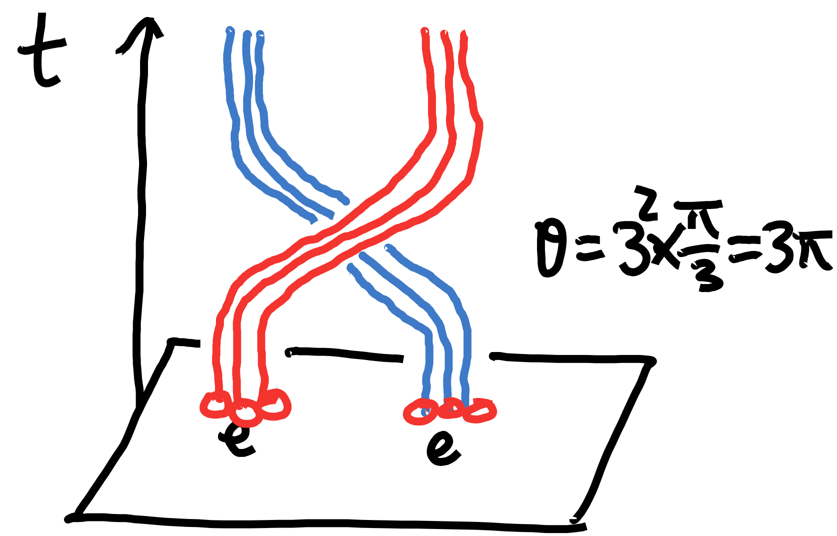


Fractional statistics

- The quasiparticle with fractional charge $e/3$ and flux hc/e also has fractional statistics.
- Two particles exchanging position by "braiding" leads to an Aharonov-Bohm phase $\theta = \pi/3$
- Fractional statistics is an intrinsic property of topological order

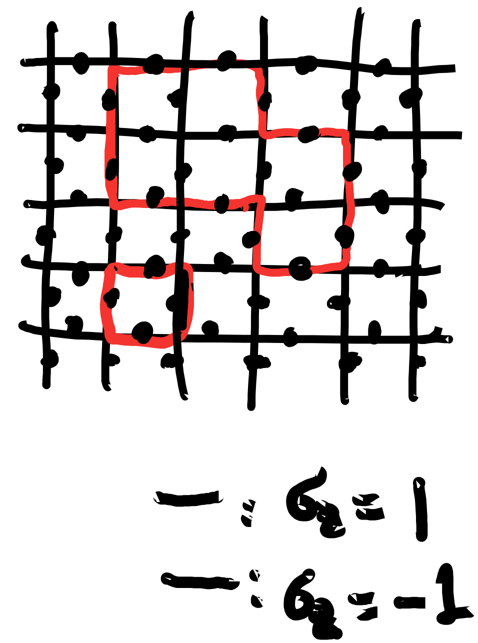


Quasiparticle braiding



electron braiding

Example 2: Toric code



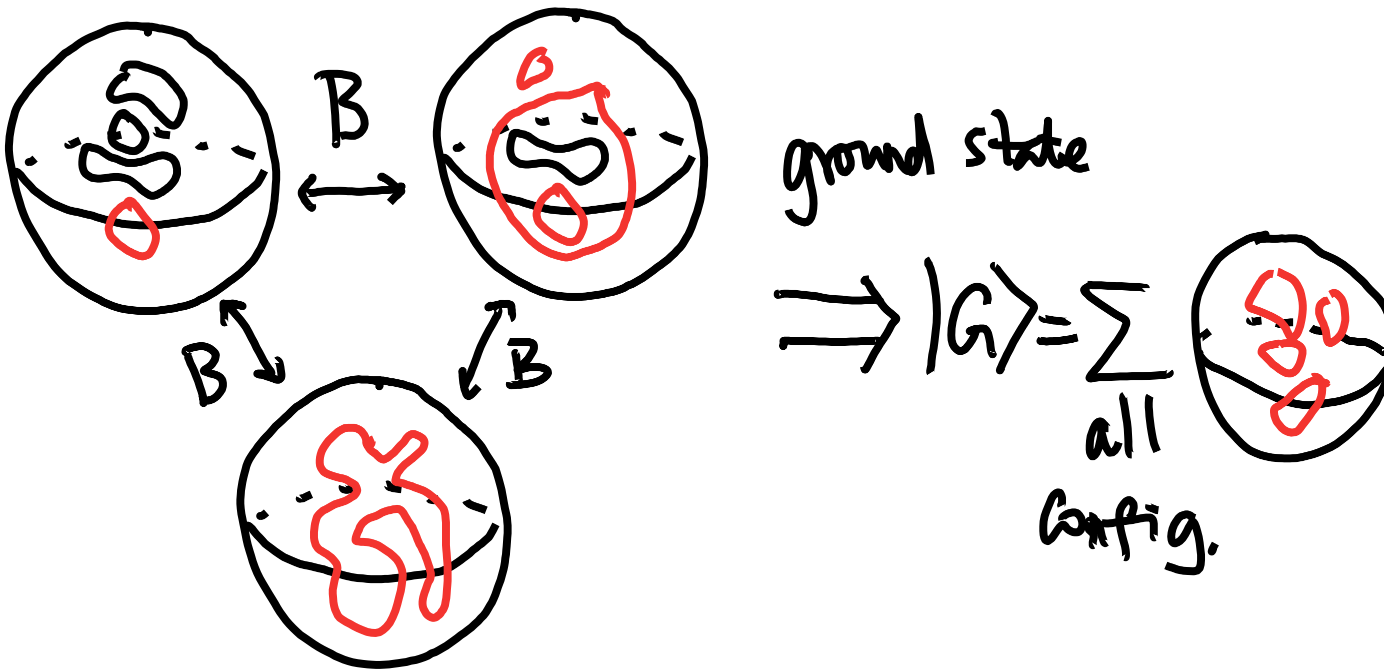
- A simple model of topological order
  (Kitaev 03', Wen 04')
- Spin ½ defined on links of a square lattice
- $H=-A\prod_{+}\sigma^{z}_{ij} - B\prod_{\square}\sigma^{x}_{ij}$
- $A>0$, $B>0$
- Ground state satisfies the Gauss law
  $\prod_{+}\sigma^{z}_{ij}=1$
- Ground state is a sum over **closed** loop configurations of $\sigma^{z}=-1$.
- The model has topological order.

Topological order of the toric code model

- Topological ground state degeneracy

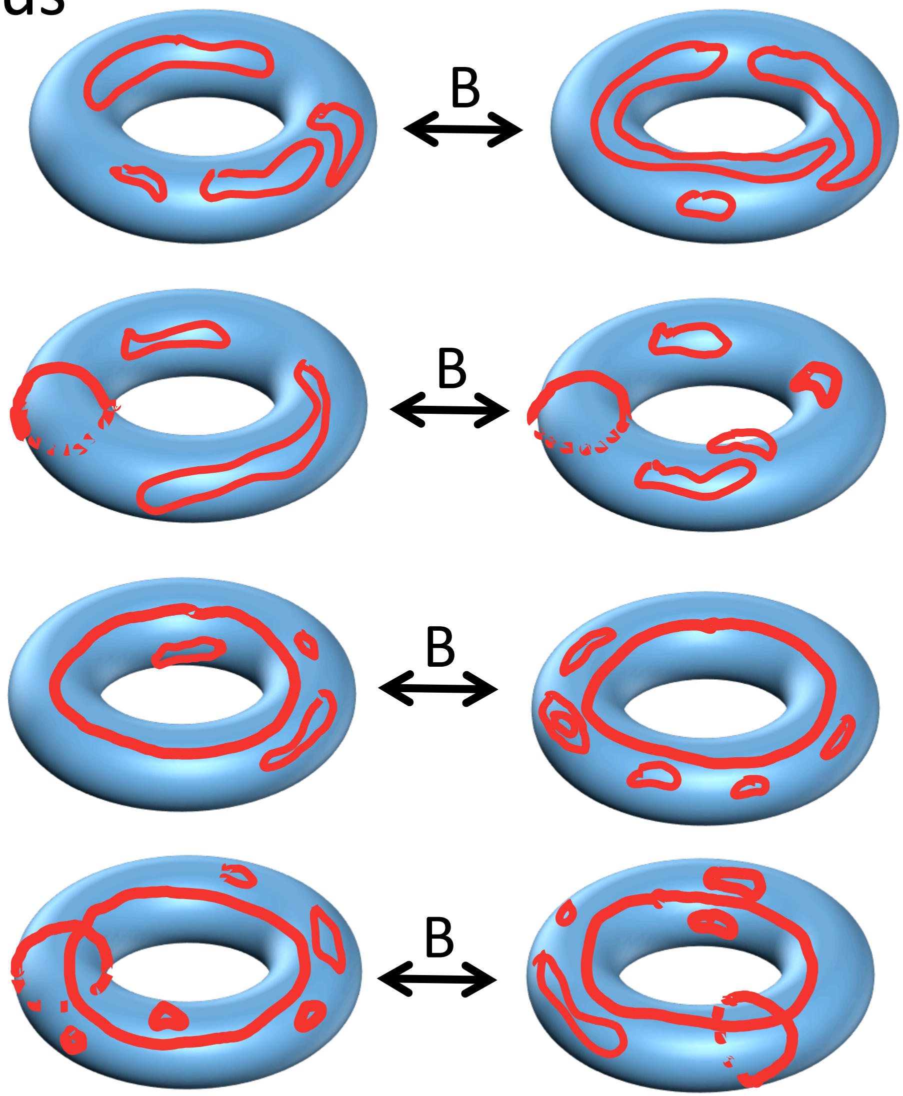
$$H = \underbrace{-A\prod_{+}\sigma^{z}_{ij}}_{H_A}\ \underbrace{-B\prod_{\square}\sigma^{x}_{ij}}_{H_B}$$

- Sphere



Topological order of the toric code model

- Torus



Not all configurations can be coupled by the Hamiltonian.

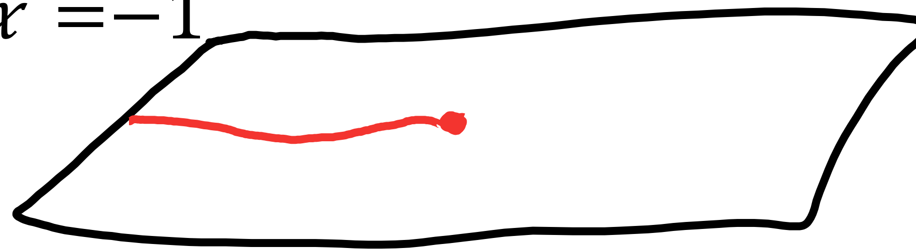
➔ There are 4 ground states

➔ Ground states can be labeled by flux in the two directions

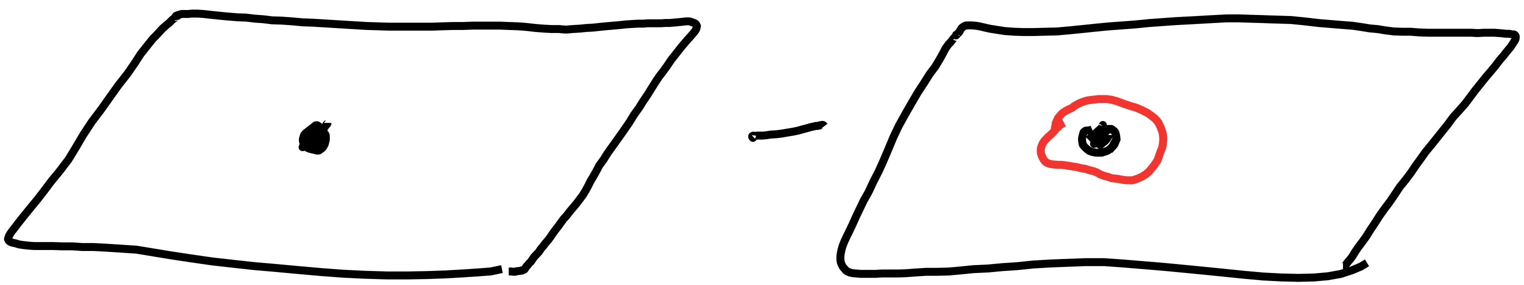
(0,0), (0,π), (π,0), (π,π)

Topological order of the toric code model

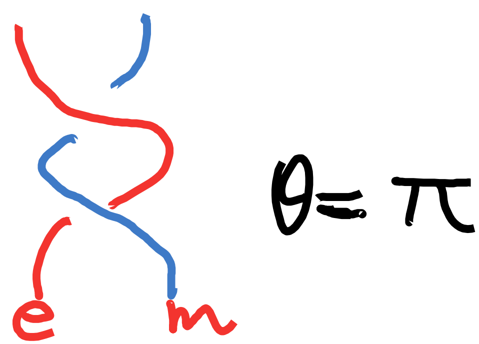
- Fractionalized excitations
- Charge e $\prod_{+} \sigma^{x} = -1$



- Flux m $\prod_{\square} \sigma^{z} = -1$



- Braiding statistics

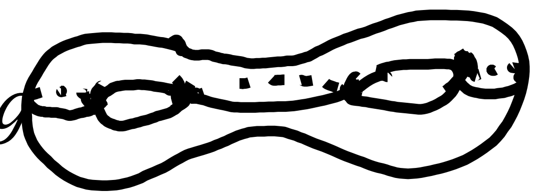


Toric code model and superconductors

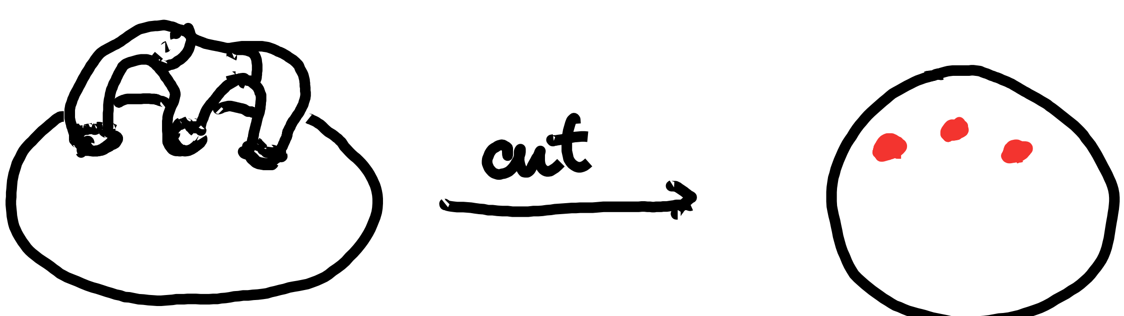
- The toric code model actually is very similar to a two-dimensional superconductor
- *If a 2D superconductor has a finite penetration depth*, it will be equivalent to a toric code model.
- $e \times m$—electron
- m—vortex with flux $hc/2e$.
- Actual 2D superconductor has a divergent vortex energy, which is why it's not strictly a topologically ordered state.

Generic features of topologically ordered states

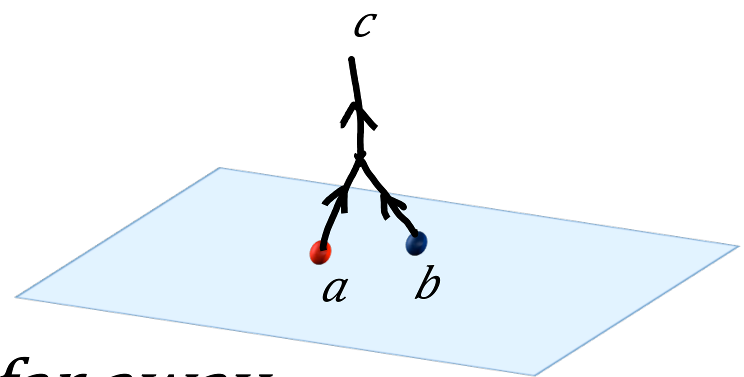
- From the two examples, we can summarize the generic features of topologically ordered states
- **Topological ground state degeneracy** determined by genus
  --Laughlin state $3\uparrow g$, Toric code $4\uparrow g$



- Excitations with fractional statistics
- Fractionalized excitations can be obtained by cutting a torus into a sphere with two punctures. Similar for higher-genus surfaces.



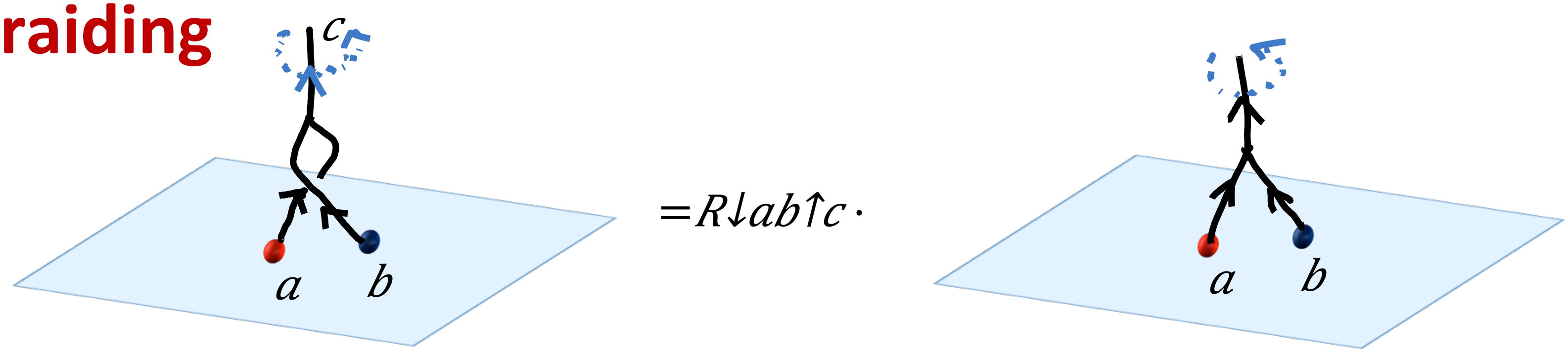
Generic features of topologically ordered states



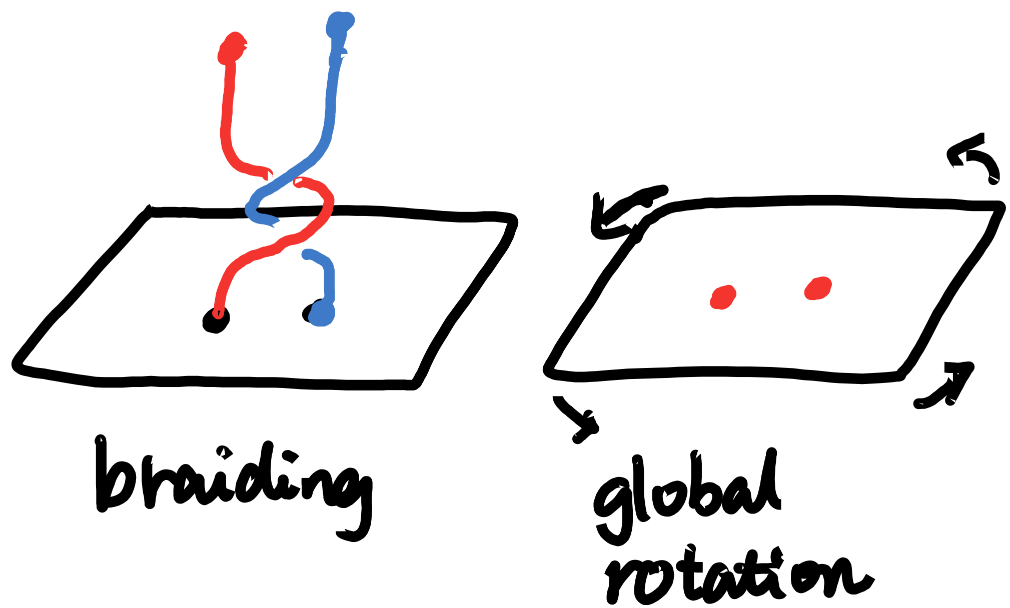
- **Fusion rule of particles**
- *Two particles together must look like a single particle from far away.*
- $a\times b=N_{ab}^{c}\, c$
- Laughlin state: $a_n \times a_m = a_{n+m}$, $n,m=0,1,2$
- Toric code: $e\times e=1$, $m\times m=1$,
  $e\times m=\psi$, $\psi\times\psi=1$
- ψ is a bound state of e,m which is a fermion. (like a superconducting quasiparticle)

Key properties of topologically ordered states

- **Braiding**



- Braiding phase may depend on the fusion channel of a,b. In general it's denoted as R^{ab}_{c}.
- **Paradox:** With only two particles, what's the difference between braiding and global rotation?



Topological spin of quasiparticles

- The difference comes from the **spin** of each particle.
- Braiding phase ＝ global rotation − spin of each particle
  ＝ spin of the fusion − spin of each particle

braiding

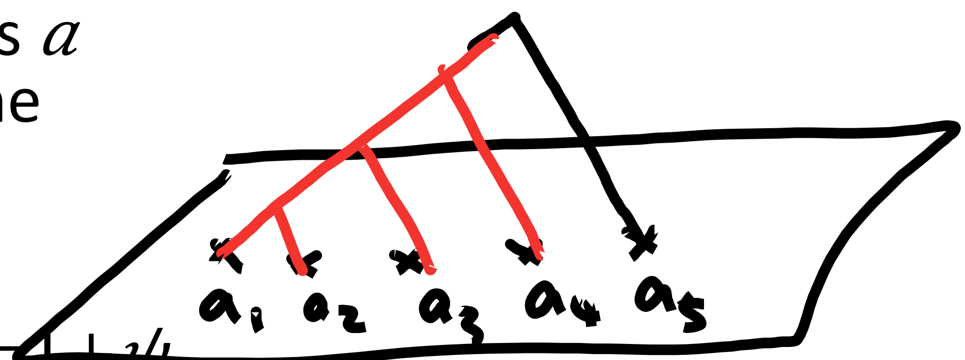
rotation

$$R_{ab}^{c} R_{ba}^{c} = e^{i2\pi(h_a + h_b - h_c)}$$

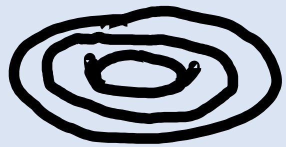
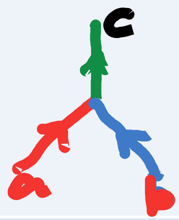
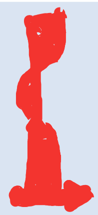
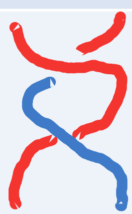
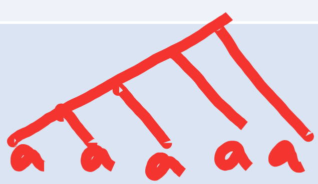
- Braiding is determined by spin of particles.
- Laughlin state $h_n = n^2/6$, toric code $h_e = h_m = 0$, $h_\psi = 1/2$.

Non-Abelian topologically ordered states

- The two examples we gave are "Abelian" topologically ordered states. The fusion of particles are definite, $a \times b = c$
- There are non-Abelian states in which particles have multiple fusion channels.
- In non-Abelian states, there is a large Hilbert space for given number of particles.
- The dimension of N particles a is $\simeq d_a^N$, d_a is called the **quantum dimension** of a
- Simplest example:
- Majorana zero modes $\sigma \times \sigma = 1 + \psi$,
- Two zero modes can fuse into a fermion occupied state or non-occupied state.
- Quantum dimension $d_\sigma = \sqrt{2}$



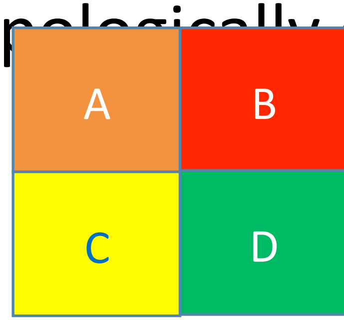
Summary of key properties of topological order

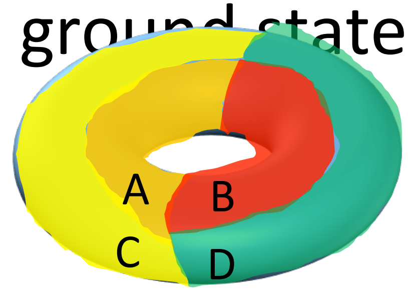
Properties	interpretation	Laughlin 1/3 state			Toric code			
Torus ground state degeneracy		3			4			
Quasiparticle fusion rule		$a_n \times a_m = a_{n+m}$, $n,m = 0,1,2 \bmod 3$			$e \times m = \psi$ $e \times e = 1$ $m \times m = 1$			
Spin of particles		0	1/6	2/3	0	0	0	1/2
Braiding statistics		$R_{nm}^{n+m} = nm\pi/3$			$R_{em}^{\psi} R_{me}^{\psi} = -1$			
Quantum dimension		1	1	1	1	1	1	1

Part II: Entanglement measures of topological order

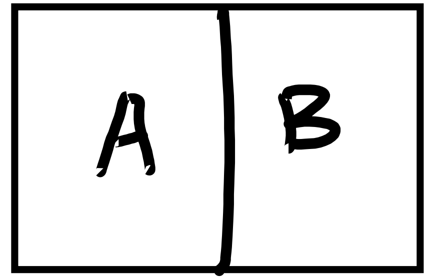
Overview about quantum entanglement

- General definition: Entanglement is a property of composite quantum system where the joint state cannot be written as a product of states of its component systems. (from www.quantiki.org)
- Simplest example: An EPR pair $|\uparrow\rangle_1 |\downarrow\rangle_2 - |\downarrow\rangle_1 |\uparrow\rangle_2$
- **Topologically ordered states are intrinsically related to quantum entanglement**
- Different topological ground states look identical in each part of a torus, but look different on the whole torus.➔Topologically ordered ground states must be entangled





Measures of quantum entanglement



- Reduced density matrix
- A state of a system with two partitions
  $|\psi\rangle = \sum_{nm} \psi_{nm} |n\rangle_A \otimes |m\rangle_B$,
- The average value of an operator O_A acting on A is
  $\langle\psi|O_A|\psi\rangle = \sum_{n,n'} \langle n'_A|O_A|n\rangle_A \sum_m \psi_{nm} \psi_{n'm}^*$
- Reduced density matrix
  $\rho_{nn'} = \sum_m \psi_{nm} \psi_{n'm}^*$
  determines expectation values of all O_A
- In short, $\rho = tr_B (|\psi\rangle\langle\psi|)$

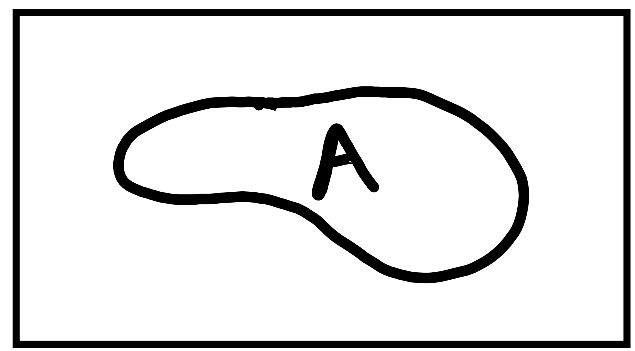
Entanglement entropy and entanglement spectrum

- The von Neumann entanglement entropy $S=-tr(\rho\log\rho)$
- $S=0$ if and only iff $\rho=|\psi\rangle\langle\psi|$ is a pure state without entanglement.
- For a spin in EPR pair, $\rho=1/2\ I$, $S=\log 2$
- Entanglement spectrum (Li&Haldane '08): eigenvalue spectrum of ρ $\mathrm{eig}(\rho)=\{\lambda_1,\lambda_2,\ldots\lambda_n\}$
- Entanglement spectrum determines the entanglement entropy $S=-\sum_n \lambda_n \log\lambda_n$ and all other bipartite entanglement properties
- Many more entanglement measures can be defined for more than two partitions

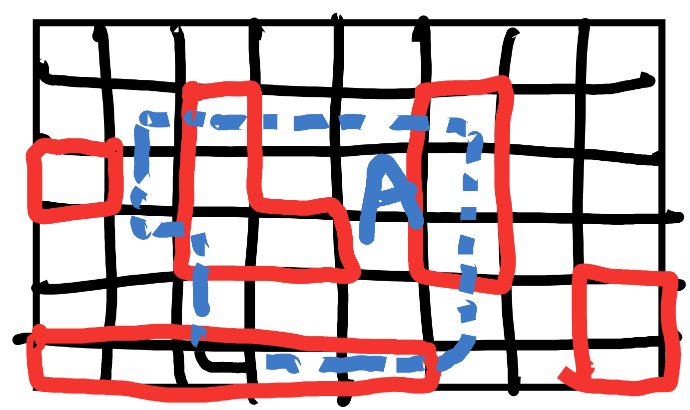
Entanglement measure I: Topological entanglement entropy

- A universal subleading term of the entanglement entropy in a topological state (Levin&Wen '06, Kitaev&Preskill '06)
- $S_A = \alpha L_A - S_{topo}$

  boundary area

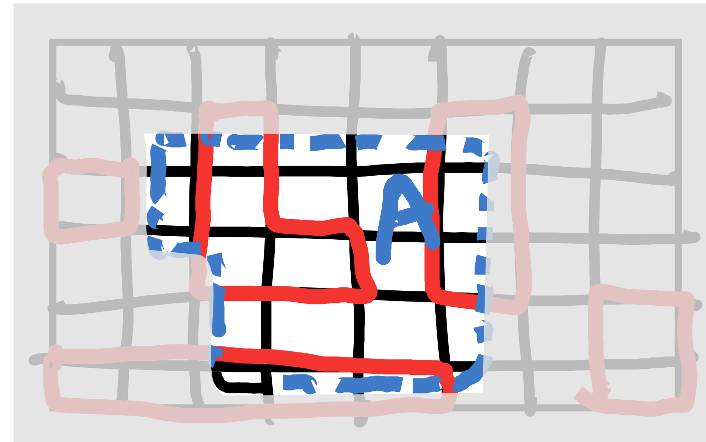


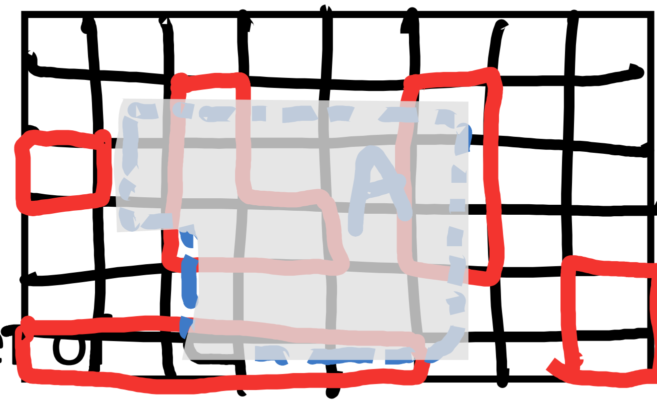
- $S_{topo} = \log \sqrt{\sum_i d_i^2}$
- Example: Toric code



Topological entanglement entropy

- Entanglement comes from the matching between the configurations in A and its complement.
- Locally, each link crossing the boundary contributes one qubit of entanglement
- Naively, $S = L_A \log 2$
- Actually, not all links are independent, due to the Gauss law $\prod_{\partial A} \sigma_z = 1$. Total number of degree of freedom $L_A - 1$
- $S = (L_A - 1)\log 2 \Rightarrow S_{topo} = \log 2$

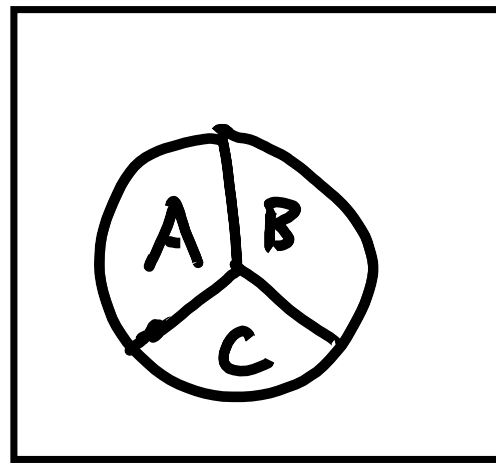




Topological entanglement entropy

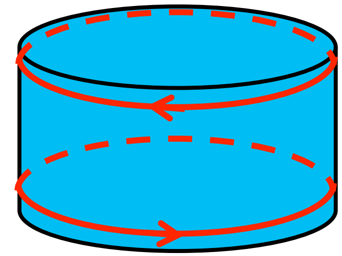
- In a finite size system it's difficult to do a fitting and get S_{topo}
- Alternatively, some combinations of entanglement entropies can be used to cancel area law term and obtain S_{topo}
- For example (Kitaev&Preskill '06)

$$S_{topo} = S_A + S_B + S_C - S_{AB} - S_{AC} - S_{BC} + S_{ABC}$$

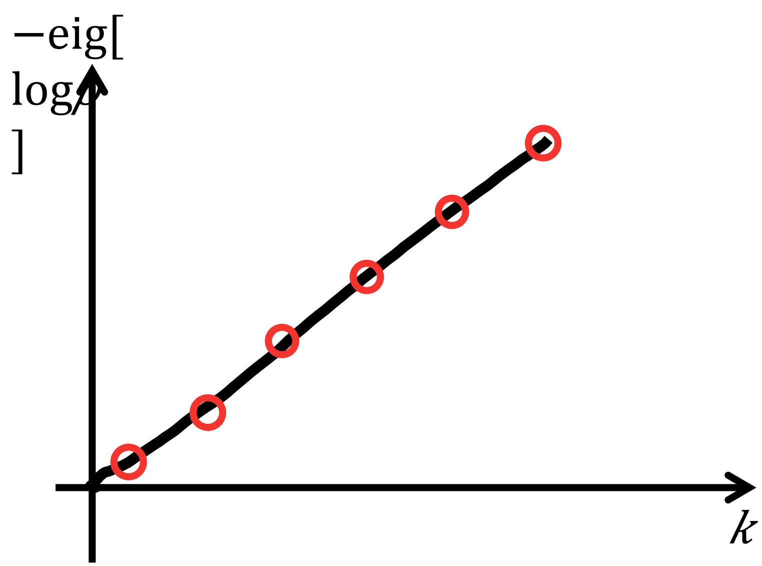


- Topological entropy probes a topological order in a single ground state
- In general it does not determine the topological order. More measures are needed.

Entanglement measure 2: entanglement spectrum

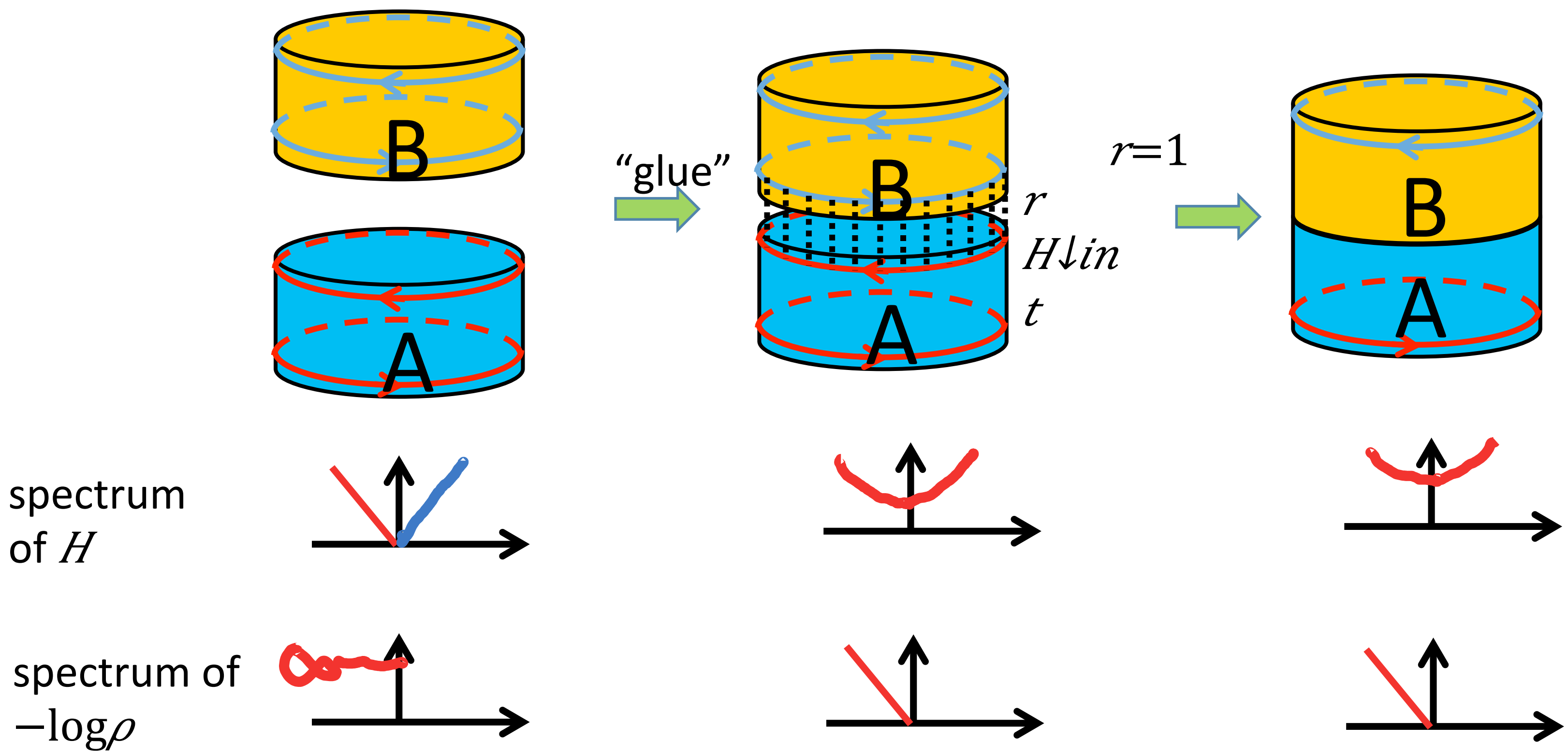


- Some topologically ordered states such as fractional quantum Hall state have chiral edge states.
- Similar edge states appear in the entanglement spectrum (Li&Haldane ‘08, Qi, Katsura&Ludwig ‘11)
- $\rho_A \simeq e^{-\beta H_{edge}}$

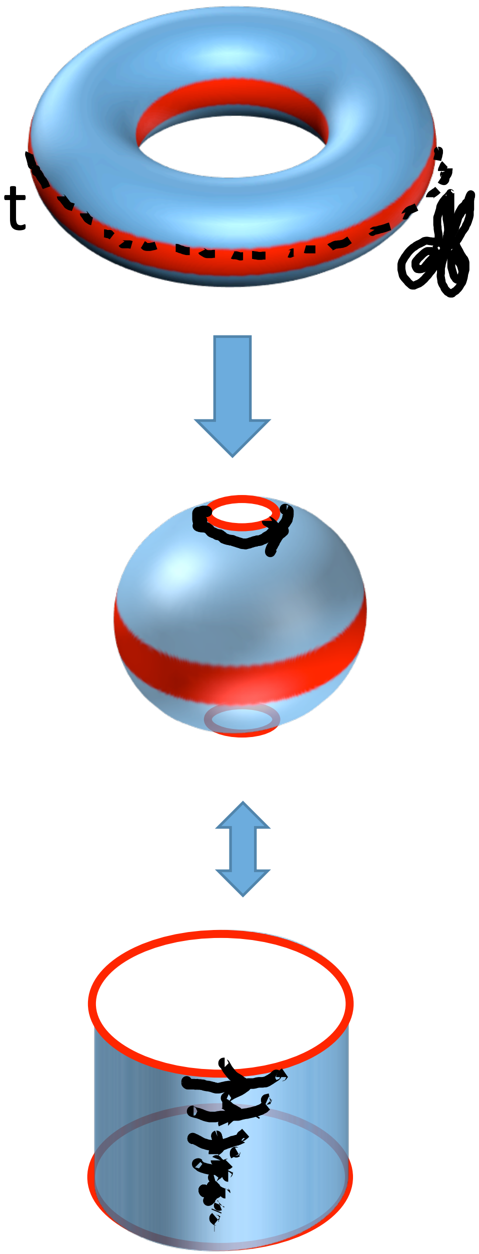


Entanglement measure 2: entanglement spectrum

- Physical reason: gapless edge states are coupled and removed from low energy spectrum.
- As a price to pay, they got entangled and shows up in the entanglement spectrum.



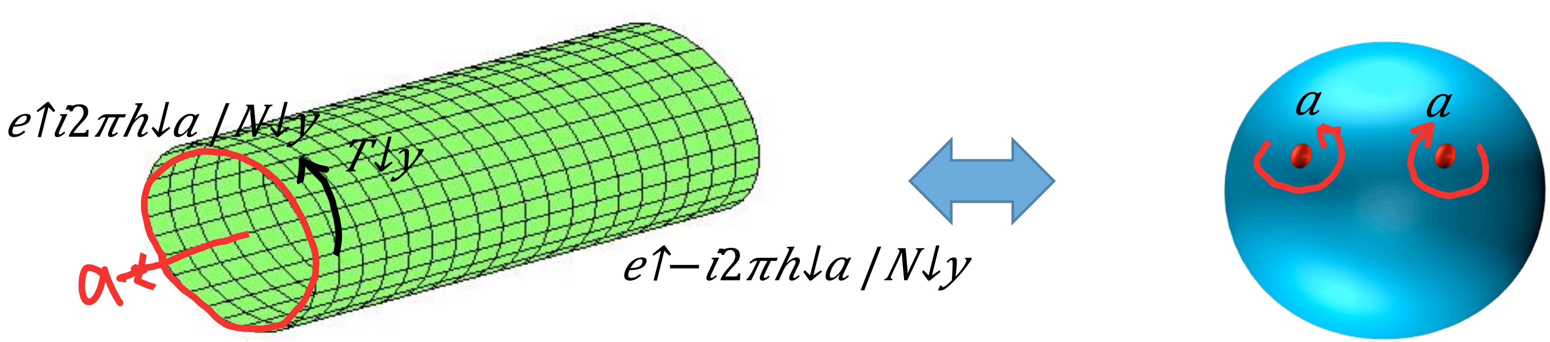
Entanglement measure 3: momentum polarization



- Topological entanglement entropy does not directly probe the topological spin of quasiparticles
- To probe the spin of particle, we need to twist a particle
- Twisting a particle is equivalent to twisting half of a cylinder
- We want to measure the Berry's phase obtained in this process

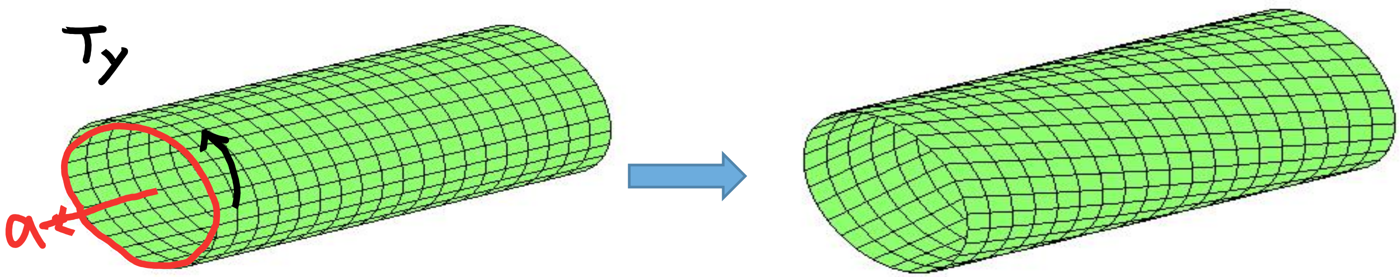
Momentum polarization

- Consider a lattice model on the cylinder, with lattice translation symmetry T_y ($T_y^{L_y}=1$)
- For a state with quasiparticle a in the cylinder, rotating the cylinder is equivalence to spinning two quasi-particles to opposite directions.
- A Berry's phase $e^{i2\pi h_a/L_y}$ is obtained at the left edge, which is cancelled by an opposite phase at the right.
- Total momentum of the left (right) edge $\pm 2\pi h_a/L_y$
  ➔ Momentum polarization $P_M = 2\pi h_a/L_y$

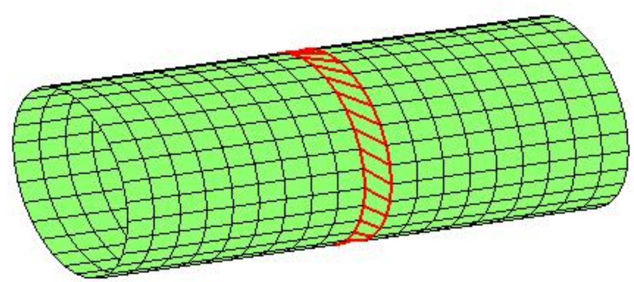


Momentum polarization

- Viewing the cylinder as a 1D system, the translation symmetry is an internal symmetry of 1D system, of which the edge states carry a projective representation.
- Ideally we want to measure

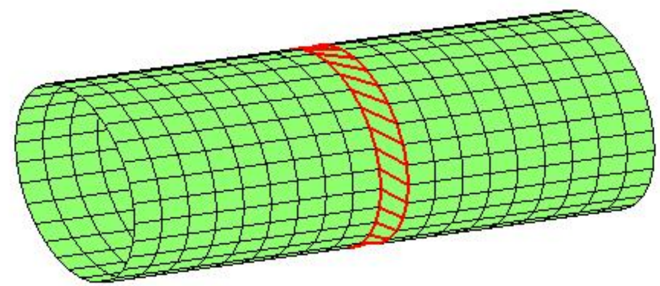


- Difficult to implement. Instead, define discrete translation T_y^L . Translation of the left half cylinder by one lattice constant



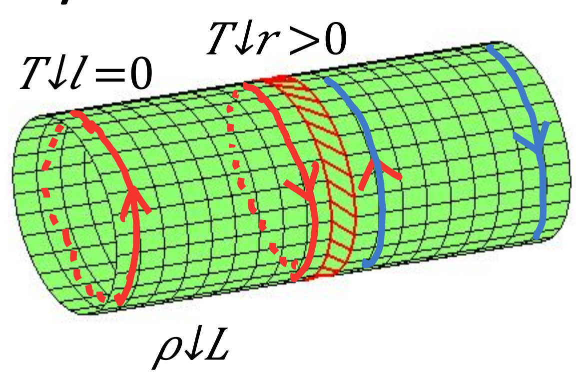
Momentum polarization

- Naive expectation: $T_y^L |G_a\rangle \sim e^{i 2\pi/L_y \, h_a} |G_a\rangle$ contributed by the left edge. However the mismatch in the middle leads to excitations and makes the result nonuniversal.
- **Actual result**: $\langle G_a | T_y^L | G_a \rangle = \exp[i 2\pi/L_y \,(h_a - c/24) - \alpha L_y]$
- The phase part has a universal subleading term
- α is independent from topological sector a
- c: chiral central charge of edge state
- Laughlin state $c=1$
- Toric code $c=0$
- Even if we don't know which sector is trivial $|G_1\rangle$, h_a can be determined up to an overall constant by diagonalizing $\langle G_n | T_y | G_m \rangle$.



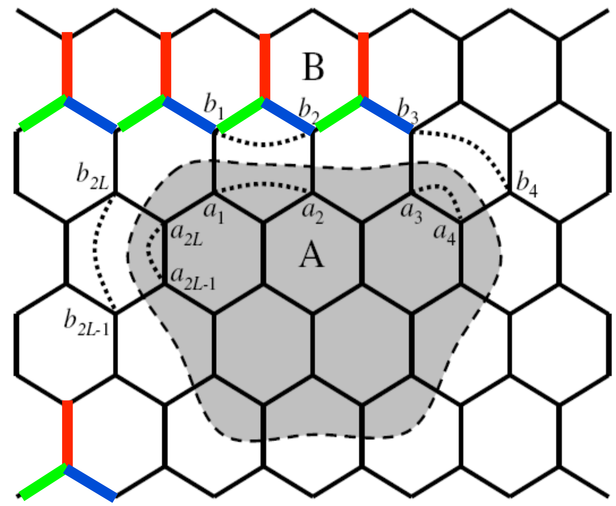
Computation of momentum polarization

- Twist T_y^L only acts on the left half system
- ➔ $\lambda_a = \langle G_a | T_y^L | G_a \rangle = tr(\rho_L T_y^L)$ is determined by the reduced density matrix of left half cylinder
- Momentum polarization λ_a is determined by edge states in the entanglement spectrum.
- **Analytic calculation of λ_a**: Using the fact that the entanglement density matrix $\rho_A = \exp[-\beta H_{edge}]$ and H_{edge} is a conformal field theory.
- ρ_L describes a cylinder with different temperature on two boundaries.
- Only right boundary has finite "temperature" due to entanglement with the other half.

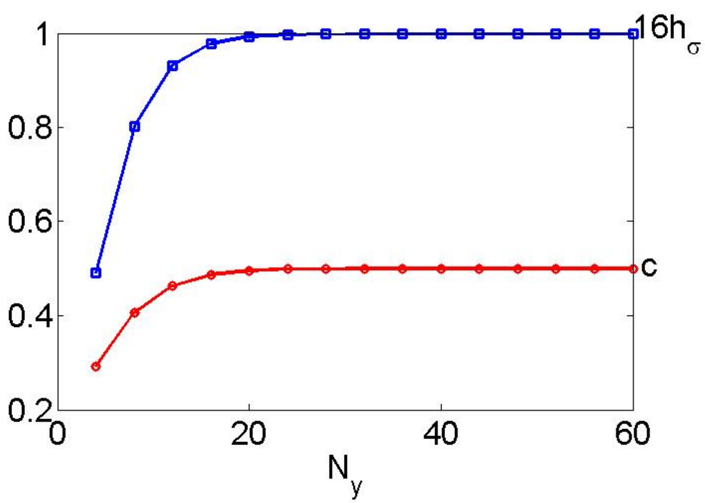


- **Numerical computation of λ_a**
- 1\. Kitaev honeycomb model (Kitaev '06). Can be calculated by mapping to free fermions coupled to Z_2 gauge field

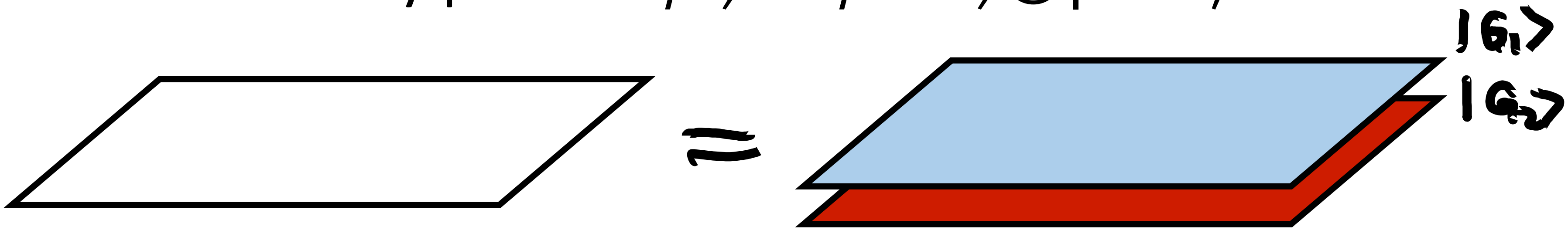
$$H=-\sum_{x-link} J_x \sigma_i^x \sigma_j^x -\sum_{y-link} J_y \sigma_i^y \sigma_j^y - \sum_{z-link} J_z \sigma_i^z \sigma_j^z$$



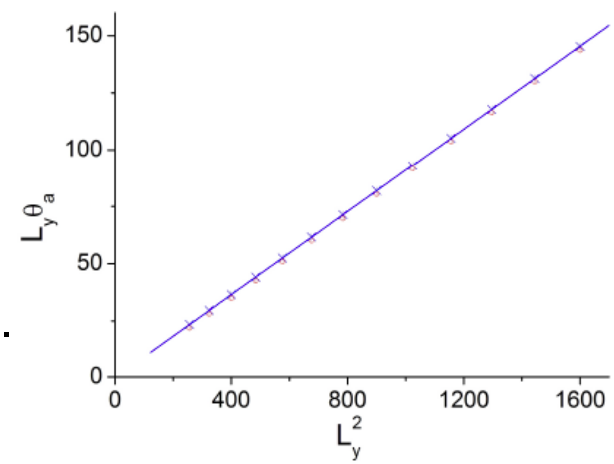
- Results agree with the expectation $c=1/2,\ h_\sigma=1/16,\ h_\psi=1/2$



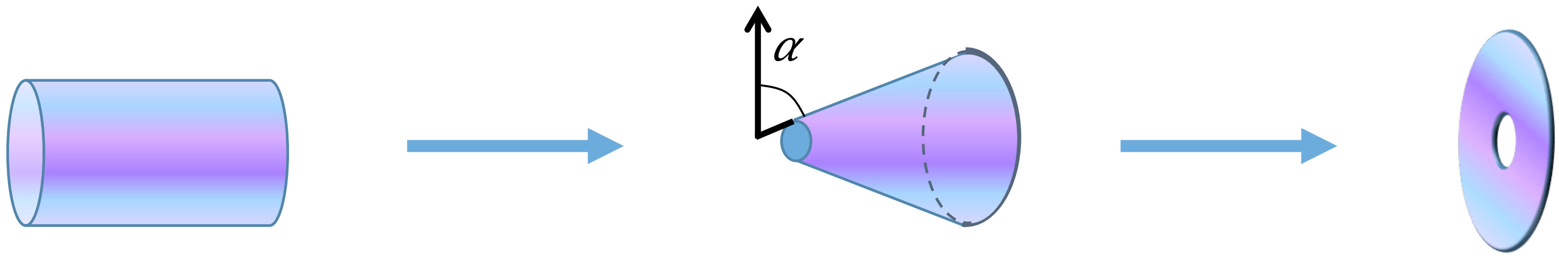
- **Numerical computation of λ_a**
- 2. Fractional Chern insulators (FCI, i.e. lattice fractional quantum Hall states)
  Similar to Laughlin state, FCI ground states can be constructed by partons $|G\rangle = P|G_1\rangle \otimes |G_2\rangle$



- Such wavefunctions can be studied by Monte Carlo.
- $\lambda_a = |\lambda_a| e^{i\theta_a}$
- Fitting
  $\theta_a L_y = -\mathrm{Im}\,\alpha L_y^2 + 2\pi(h_a \cdot$
  (Tu&Zhang&Qi,'12, Zhang&Qi, '13)



Momentum polarization in more generic geometries

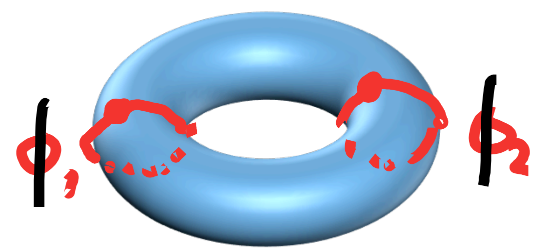


$(h_a - c/24)2\pi/L_y$ $\quad$ $(h_a - c/24 \sin^2 \alpha)2\pi/L_y$ $\quad$ $h_a 2\pi/L_y$

- By studying the momentum polarization on a cone and varying the cone angle, the central charge c contribution can be determined
- c contribution is geometrical, while h_a is topological.
- Verified by momentum polarization of a cMPS state for the Pfaffian wavefunction (Mong, Zaletel, Qi)

Entanglement measure 4: Topological uncertainty relation

- Topological sector can be measured by quasiparticle paths around the torus
- For Laughlin state, taking $q=e/3$ particle around the torus measures the flux.

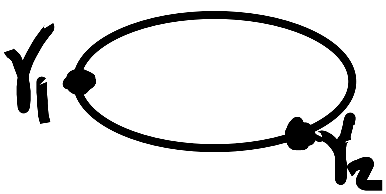


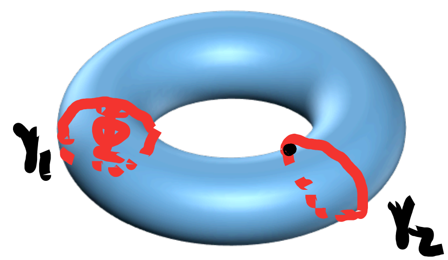
- The measurement can be done at any loop ➔ A long range order of string order parameter.
- Long-range correlation between loop operators $\langle \phi(r_1)\phi(r_2)\rangle = 1$
- Similar to classical order in a ferromagnet $\langle S(r_1)S(r_2)\rangle = M^2$

Chao-Ming Jian, Isaac Kim & XLQ, in preparation

Comparison between conventional order and topological order

- Spontaneous symmetry breaking leads to classical long-range order
- $H=-J\sum_i \sigma_i^z \sigma_{i+1}^z$ Ising model
- Ground states $|\uparrow\uparrow\ldots\uparrow\rangle$, $|\downarrow\downarrow\ldots\downarrow\rangle$
- Comparison with topological order





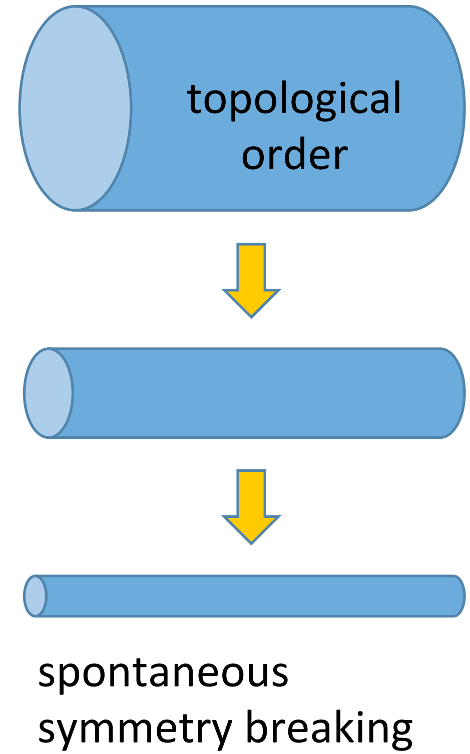
Spin eigenstate $|\uparrow\uparrow\ldots\uparrow\rangle$ and $|\downarrow\downarrow\ldots\downarrow\rangle$

Flux eigenstate $|\phi\rangle$, $\phi=0, 2\pi/3, 4\pi/3$

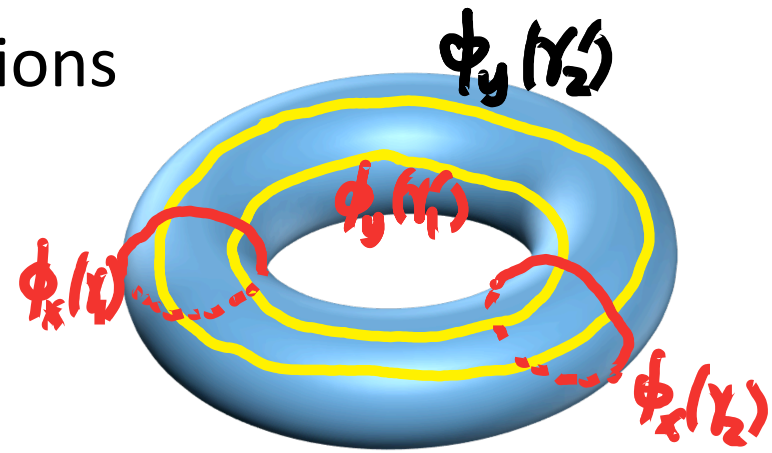
Spin correlation $\langle\sigma_z(r_1)\sigma_z(r_2)\rangle=1$

Flux correlation between two loops $\langle\phi(r_1)\phi(r_2)\rangle=1$

Comparison between conventional order and topological order

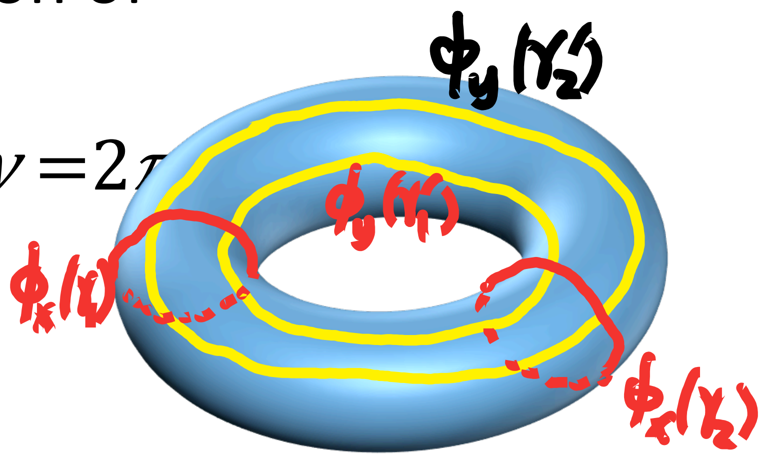


- Topological order is like a conventional order after "dimensional reduction" to lower dimension
- Is that it? What's the key difference between topological order and classical long-range order?
- A torus can be reduced to 1D in two different ways
- Two kinds of long-range correlations for loops ϕ_x and ϕ_y.
- Each looks like a long-range order but they don't commute.

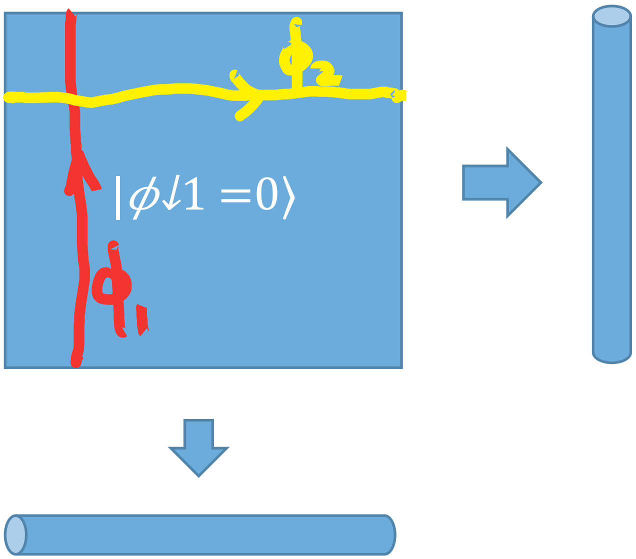


Non-commuting long-range correlations between loop operators

- Measuring ϕ_x requires to take a quasiparticle going around loop X.
- Quasiparticle carries charge $e/3$ and flux $2\pi/3$ (i.e. hc/e)
- ➔ Flux in loop Y is changed by $2\pi/3$.
- $e^{i\phi_x} e^{i\phi_y} = e^{i\phi_y} e^{i\phi_x} e^{i2\pi/3}$
- Eigenstates of ϕ_x is superposition of ϕ_y eigenstates
- $|\phi_x = 0\rangle = 1/\sqrt{3} \sum_{n=0}^{2} |\phi_y = 2\pi$



Non-commuting long-range correlations between loop operators



A “Schroedinger’s cat state”

$\frac{1}{\sqrt{3}}(|\uparrow\rangle + |\searrow\rangle + |\swarrow\rangle)$

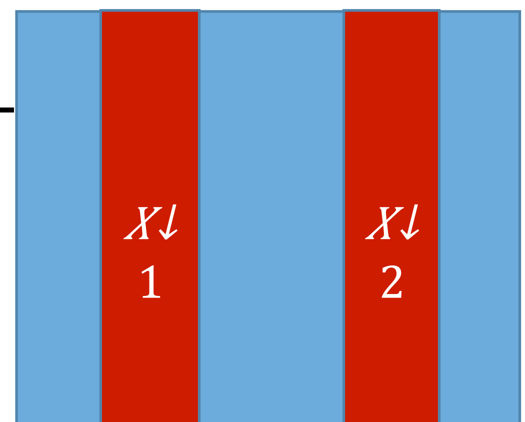
A spontaneous symmetry breaking state

$|\uparrow\rangle$

- Lesson: Topological order can be understood as long-range order of **non-commuting** loop operators

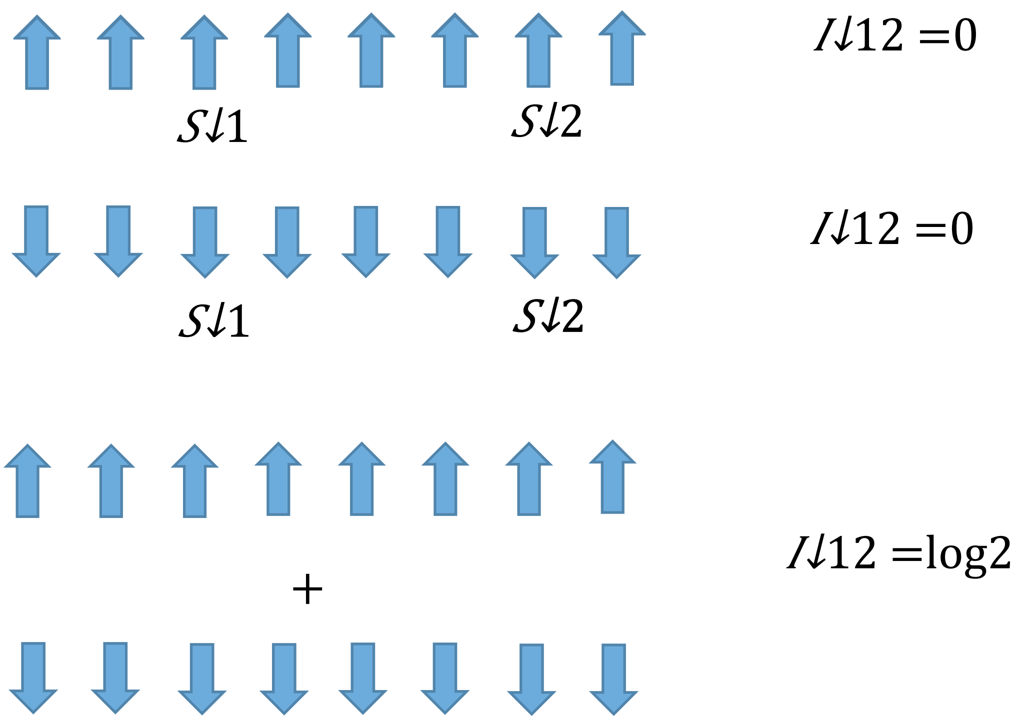
Topological uncertainty relation

- Non-commuting operators such as $[x,p]=i$ lead to Heisenberg's uncertainty relation.
- For a topological order, loop operators on the torus cannot be simultaneously diagonalized
- We can define a quantum entanglement measure using this intuition.
- Define mutual information between two regions on a torus
- $I_{X_1 X_2} = S_{X_1} + S_{X_2} -$
- Mutual information measures correlation between the two regions.

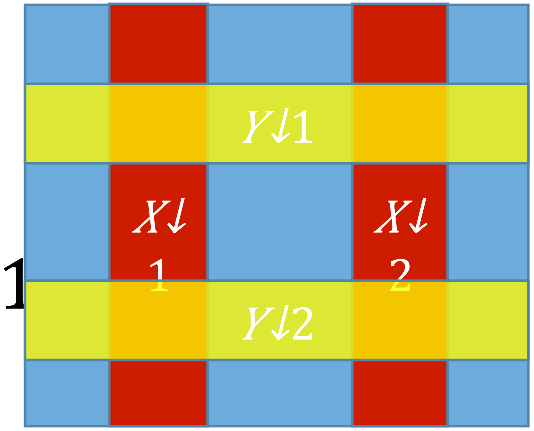


Mutual information measures long-range order

- An EPR pair $|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle$, $S_1 = S_2 = \log 2$, $S_{12} = 0$, $I_{12} = 2\log 2$ is maximal.
- For classical long-range order



Topological uncertainty relation



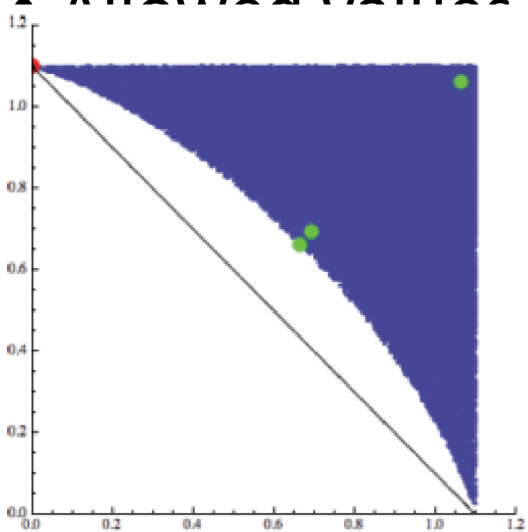
- For topological order, a state $|\phi_x = 0\rangle$ has
- $I_X = I_{X_1 X_2} = 0$, $I_Y = I_{Y_1}$ …
- Alternative, if we take $|\phi_y = 0\rangle$, it has $I_X > 0$, $I_Y = 0$.
- I_X and I_Y cannot simultaneously vanish.
- $I_X + I_Y$ has a lower bound, as a consequence of the uncertainty relation (Jian, Kim & XLQ '15)
- $I_X + I_Y \geq -2\log \max_{n,m} |S_{nm}|$
- $S_{nm} = \langle n_X | m_Y \rangle$ the transformation matrix between the two basis, also known as the modular S-matrix.

Topological uncertainty relation

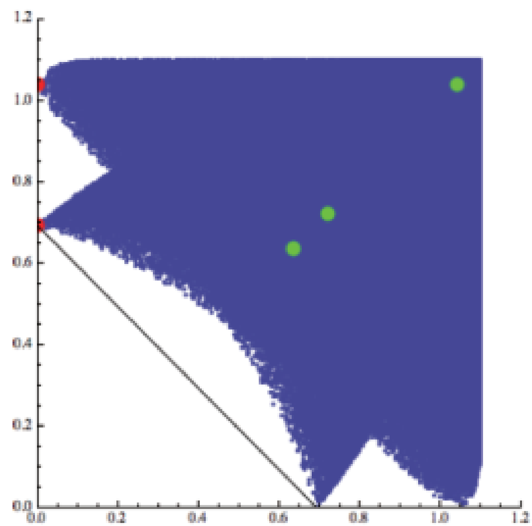
- For a generic ground state $|\Psi\rangle = \sum_n a_n |n_X\rangle = \sum_n b_n |n_Y\rangle$,

  $I_X = -\sum_n |a_n|^2 \log |a_n|^2$, $I_Y = -\sum_n |b_n|^2 \log |b_n|^2$.

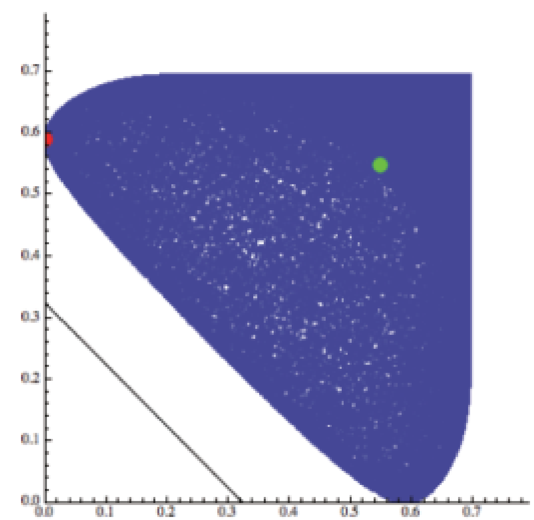
- Allowed values of (I_X, I_Y) f… examples:



Laughlin 1/3



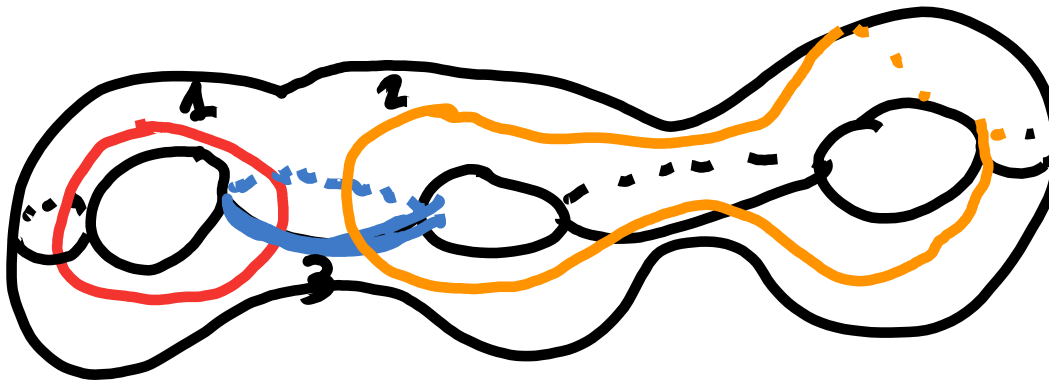
$SU(2)_2$



Fibonacci

More general topology

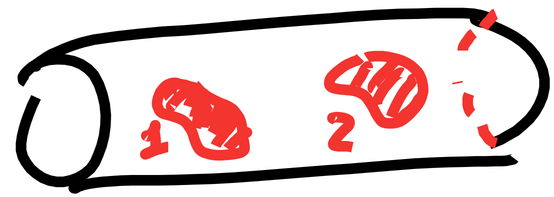
- On a more generic manifold, any two loops L_1 and L_2 with nontrivial intersection has non-commuting loop operators



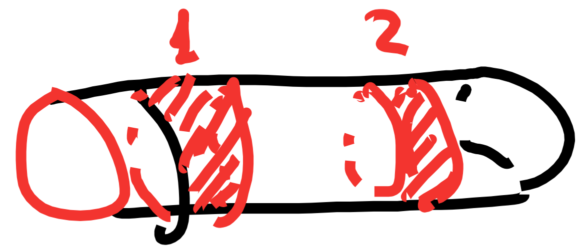
- $\min(I_1 + I_3) > 0, \min(I_2 + I_3) > 0, \min(I_1 + I_2) = 0$

Summary of topological uncertainty relations

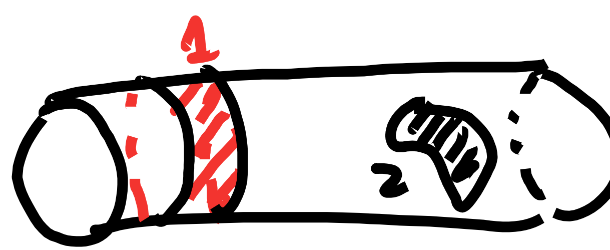
- In general, we can compute mutual information between regions in the system



- Topological order means
  1) Contractible regions have no mutual information
  2) Non-contractible regions generically have nonzero mutual information
  3) When two pairs of loops ($X1$, $X2$ and $Y1$, $Y2$) have nonzero intersection, $I\downarrow X + I\downarrow Y$ have finite lower bound.



- This approach provides a general characteristics of topological order which can be generalized to higher dimensions.



- A direct measure of "long-range entanglement".

Summary

- Topologically ordered states are states of matter with ground state degeneracy, fractional statistics etc.
- Topological order is intrinsically related to quantum entanglement.
- Topological order is difficult to probe, and quantum entanglement provides helpful characteristics.
- Different quantum entanglement measures can be defined to characterize topologically ordered states, such as topological entanglement entropy, entanglement spectrum, momentum polarization and topological uncertainty relation.
- Many more open questions in higher dimensions.

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Thanks!