Topological nonsymmorphic crystalline insulators and superconductors

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Outline

- Introduction, symmetry and topology in the classification of states of matter
- Topological mirror insulators and superconductors
- Nonsymmorphic symmetry and topological phases
  - Topological non-symmorphic superconductors
  - Topological non-symmorphic insulators
- Conclusion and outlook
Landau theory: states of matter are classified by symmetry.

E.g. a solid is different from a gas because it breaks translational symmetry.
Quantum Hall Effect

- The quantum Hall effect: topological states, which is beyond Landau theory

\[ \sigma_H = C \frac{e^2}{h} \]

where \( C = 1, 2, \ldots, n \)

TKNN (1982)

Klaus von Klitzing (1980)
Quantum Hall effect

- The quantum Hall effect and chiral edge states
Topological states of matter

- Topological states for free fermions
  - Topological states cannot be adiabatically connected to normal states without gap closing even though sharing the same symmetry. E.g. Quantum Hall state with Landau gaps.
  - Topological states have edge/surface modes at the boundary (surface or interface). E.g. Chiral edge states of the QH effect
Question: can we have more topological phases beyond the quantum Hall state?

For a long time, people thought the QH state is the only example.

Recent development has shown more topological states when there are additional symmetries.

- Topological insulators: insulating bulk gap and helical edge states which are protected by time reversal symmetry.
- Topological superconductors: superconducting bulk gap and Majorana zero modes
- Topological crystalline insulators and superconductors
- More topological states due to interaction.

Qi and Zhang, RMP (2011), Hasan and Kane, RMP (2010).
Topological insulators

Quantum Hall state
Magnetic fields

Quantum spin Hall state
(2D topological insulators)
Spin-orbit coupling

Break TR \( \sigma_H \neq 0 \)

Preserve TR \( \sigma_H^\uparrow = -\sigma_H^\downarrow \)

Topological insulators

• Helical edge/surface states and time reversal symmetry

➢ Krammers’ theorem, time reversal can protect degeneracy of spinful fermions when $\Theta^2 = -1$.

$$E(\vec{k}, \uparrow) = E(-\vec{k}, \downarrow)$$

$$\vec{k} = -\vec{k} + \vec{G}$$
Topological insulators

- How does Kramers' degeneracy protect non-trivial surface states?
Topological insulators

- Topological insulators in 3D: an odd number of Dirac cones at one surface.
Topological insulators

• Experimental observation of helical edge states in topological insulators


Topological superconductors

- Topological superconductors and particle-hole symmetry
  - Superconducting gap in the bulk
  - Gapless excitation at the boundary with zero energy at zero momentum (Majorana fermion or Majorana zero modes).

\[ E \Delta \]

\[ k \]

\[ k_F \]

\[ C = 1, 2, \ldots, n \]

Read and Green (2000)
Topological superconductors

- Bogoliubov-de Gennes Hamiltonian and particle-hole symmetry (redundancy).

\[ \hat{H} = \frac{1}{2} \sum_k (\psi_k^+ \psi_{-k}) H_{BdG} \left( \begin{array}{c} \psi_k \\ \psi_{-k}^+ \end{array} \right) \]

\[ H_{BdG} = \begin{pmatrix} h(k) - \mu & \Delta(k) \\ \Delta^+(k) & -h^*(-k) + \mu \end{pmatrix} \]

Particle-hole symmetry

\[ CH_{BdG}C^{-1} = -H_{BdG} \]

\[ C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} K \]
Particle-hole symmetry and Majorana zero modes

Alicea (2012), Beenakker (2011)

Particle-hole partner

\[ \phi_0 = C \phi_0 \rightarrow \gamma_0^+ = \gamma_0 \]

Majorana fermions: Particle=Antiparticle

\[ \phi_{-E} = C \phi_E \rightarrow \gamma_E^+ = \gamma_{-E} \]
Topological superconductors

- Kitaev model for 1D p-wave superconductor

\[ H - \mu N = \sum_i t(\psi_i^+\psi_{i+1} + \psi_{i+1}^+\psi_i) - \mu \psi_i^+\psi_i + \Delta(\psi_i\psi_{i+1} + \psi_{i+1}^+\psi_i^+) \]

\[ \psi_i = \gamma_{1i} + i\gamma_{2i}, \quad \gamma_{ai}^+ = \gamma_{ai} \]

\[ H = 2i \sum_i t_1 \gamma_{1i} \gamma_{2i} + t_2 \gamma_{2i} \gamma_{1i+1} \]

Kitaev (2001)

Su, Schrieffer and Heeger (1979)
Topological superconductors

- Kitaev model for 1D $p$ wave superconductor

Kitaev (2001)

$$ H = 2i \sum_i t_1 \gamma_{1i} \gamma_{2i} + t_2 \gamma_{2i} \gamma_{1i+1} $$

Trivial phase

$t_1 > t_2$

Topological phase

$t_1 < t_2$

Unpaired end Majorana zero modes
Topological superconductors

- \( \text{P+ip chiral topological superconductors} \)

1d \( p \) wave TSC with 0d (bound) majorana end mode

2d \( p+ip \) TSC with 1d Majorana edge mode

Read and Green (2000)
### Topological classification

- **Schnyder-Ryu-Furusaki-Ludwig (SRFL) classification**

  *Schnyder, Ryu, Furusaki, Ludwig (2008)*

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- Conclusion and outlook
Topological states and crystalline symmetry

- Topological surface states protected by other symmetries? (beyond SRFL classification)
  - Anti-unitary, magnetic crystalline TIs.
  - Different irreducible representations, mirror TIs and non-symmorphic TSCs.
  - Non-commutation, non-symmorphic TIs, C4v TIs.
Example: topological mirror insulators.

- Two subspaces with opposite mirror parities are decoupled.

\[ m_z \psi_+ = |\psi_+\rangle \quad m_z \psi_- = |\psi_-\rangle \]

\[ m_z V m_z^+ = V \]

\[ \langle \psi_+ | V | \psi_- \rangle = \langle \psi_+ | m_z^+ m_z V m_z^+ m_z | \psi_- \rangle = -\langle \psi_+ | V | \psi_- \rangle \rightarrow \langle \psi_+ | V | \psi_- \rangle = 0 \]
Example: topological mirror insulators.

Two subspaces with opposite mirror parities are decoupled.

Mirror invariant plane

\[ m_z: (x, y, z) \rightarrow (x, y, -z) \]

\[ C = C_e + C_o = 0 \]

\[ C_M = C_e - C_o \neq 0 \]

Mirror subspace

Even

\[ C_e = 1 \]

Odd

\[ C_o = -1 \]
Topological mirror insulators

- Theoretical prediction and experimental observation of topological mirror insulators in SnTe family of materials


Can similar idea be applied to superconducting states? (Topological mirror superconductors)

Key question: will particle-hole symmetry still exist in one mirror parity subspace? Equivalently, do a state $\psi$ and its particle-hole partner $\tilde{\psi} = C\psi$ have the same mirror parity or not?

Zhang (2013), Ueno (2013)
In a mirror superconductor, particle-hole symmetry might exist in one mirror parity subspace or not, depending on gap function.

- Gap function is classified by irreducible representations of symmetry group.

\[
D(m)\Delta(k)D^T(m) = \eta\Delta(k), \quad \eta = \pm
\]

\[
\Delta(k) = V_0 \langle \psi_{-k} \psi_k \rangle
\]

- The \( \eta = - \) means that superconducting phase spontaneously breaks mirror symmetry.
Key step: mathematically, we can still define a new “mirror” symmetry operation for the BdG Hamiltonian.

\[
D(m)\Delta(k)D^T(m) = \eta\Delta(k), \quad \eta = \pm
\]

\[
D(m) = \begin{pmatrix} D(m) & 0 \\ 0 & \eta D^*(m) \end{pmatrix} \quad H_{BdG} = \begin{pmatrix} h(k) - \mu & \Delta(k) \\ \Delta^+(k) & -h^*(-k) + \mu \end{pmatrix}
\]

\[
D(m)H_{BdG}D^+(m) = H_{BdG}
\]
Physical meaning of the new “mirror” symmetry operator $D(m)$. The parity of hole band is determined by $\Delta(k)$.

$$D(m)\Delta(k)D^T(m) = \Delta(k)$$

$$D(m)\Delta(k)D^T(m) = -\Delta(k)$$
Topological mirror Superconductors

- Any eigen-state of the BdG Hamiltonian can have the definite mirror parity.

\[ \mathcal{D}(m)H_{BdG}\mathcal{D}^+(m) = H_{BdG} \]

\[ H_{BdG}\psi = E\psi, \quad \mathcal{D}(m)\psi = \delta_m\psi, \quad \delta_m = \pm \]

- The mirror parities of \( \psi \) and \( \tilde{\psi} \) are determined by \( \eta \).

\[ CD(m)C^{-1} = \eta\mathcal{D}(m) \]

\[ \tilde{\psi} = C\psi, \quad H_{BdG}\tilde{\psi} = -E\tilde{\psi}, \quad \mathcal{D}(m)\tilde{\psi} = \eta\delta_m\tilde{\psi} \]
When \( \eta = + \), \( \psi \) and \( \tilde{\psi} \) have the same mirror parity, while \( \eta = - \), \( \psi \) and \( \tilde{\psi} \) have opposite mirror parities.
**Topological mirror Superconductors**

- Different topological classifications for different types of gap functions.

\[ D(m)\Delta(k)D^T(m) = \eta\Delta(k), \quad \eta = \pm \]

**D class**

\[ \delta_m = + \quad \eta = + \]
\[ \delta_m = - \quad \eta = - \]

Zhang (2013), Ueno (2013)
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Non-symmorphic symmetry

- Non-symmorphic symmetry: glide reflection and screw axis

Screw axis

\[ C_n | \vec{\tau} \]

Glide reflection

\[ (m | \vec{\tau}) \]
Non-symmorphic materials

- 157 of 230 space groups are non-symmorphic
- Examples of non-symmorphic systems

Diamond

Iron-based superconductors
Electronic states in a non-symmorphic crystal

- Electronic states in a system with glide symmetry (glide parity)

\[ D(g)\psi_k = \delta_m e^{i\vec{k} \cdot \vec{\tau}} \psi_k = \delta_m e^{i k_x a/2} \psi_k, \quad \delta_m = \pm, \vec{\tau} = \left(\frac{a}{2}, 0, 0\right) \]

- All the bands appear in pairs.

\[ D(g)\psi_{k+2\pi/a} = \delta_m e^{\frac{i k_x a}{2} + i\pi} \psi_{k+2\pi/a} = -\delta_m e^{\frac{i k_x a}{2}} \psi_{k+2\pi/a} \]

Fang and Fu (2015)
We can follow the logic of topological mirror superconductors and identify glide parity for the states $\psi$ and $\tilde{\psi}$. (Homework???)

$$D_{\vec{k}}(g)\Delta(\vec{k})D^{T}_{-\vec{k}}(g) = \eta \Delta(\vec{k}), \quad \eta = \pm$$

$$G(g)\psi = \delta_{\eta} e^{i\vec{k} \cdot \vec{\tau}} \psi, \quad G(g)\tilde{\psi} = \eta \delta_{\eta}^{*} e^{-i\vec{k} \cdot \vec{\tau}} \tilde{\psi}, \quad \delta_{\eta} = \pm$$

$\eta = +$

$\psi: e^{i\vec{k} \cdot \vec{\tau}}(-e^{i\vec{k} \cdot \vec{\tau}})$

$\tilde{\psi}: e^{-i\vec{k} \cdot \vec{\tau}}(-e^{-i\vec{k} \cdot \vec{\tau}})$

$\eta = -$}

$\psi: e^{i\vec{k} \cdot \vec{\tau}}(-e^{i\vec{k} \cdot \vec{\tau}})$

$\tilde{\psi}: -e^{-i\vec{k} \cdot \vec{\tau}}(e^{-i\vec{k} \cdot \vec{\tau}})$
Glide parity

- Glide parities of $\psi$ and $\tilde{\psi}$ depend on the momentum

\[ \eta = + \]

\[ \vec{k} \cdot \vec{\tau} = 0 \]

$\psi$: $+$, $\tilde{\psi}$: $+$ \hspace{1cm} \text{Same}

\[ \vec{k} \cdot \vec{\tau} = \frac{\pi}{2} \]

$\psi$: $+i$, $\tilde{\psi}$: $-i$ \hspace{1cm} \text{Opposite}
Glide parity and particle-hole symmetry

- Particle-hole symmetry only exists in one line in the 2D Brillouin zone for one glide parity subspace.
- Topological invariant can be only defined in 1D line, but not in 2D plane.

\[ \eta = + \]

\[ \eta = - \]
Topological glide superconductors

- There is a $\mathbb{Z}_2$ classification for the D class, the Kitaev model.
- Our model for topological glide superconductors in a distorted square lattice.
Topological glide superconductors

- **Model Hamiltonian**

\[ h(k) = \epsilon(k)\sigma_0 + t_3\cos\left(\frac{(k_x - \phi)a}{2}\right)\cos\left(\frac{k_x a}{2}\right)\sigma_1 + t_3\cos\left(\frac{k_x - \phi)a}{2}\right)\sin\left(\frac{k_x a}{2}\right)\sigma_2 \]

\[ \sigma \text{ is for A and B sublattices} \]

**Glide symmetry**

\[ D_k(g) = e^{i\frac{k_x a}{2}}\left(\cos\left(\frac{k_x a}{2}\right)\sigma_1 + \sin\left(\frac{k_x a}{2}\right)\sigma_2\right) \]

\[ D_k(g)h(k_x, k_y)D_k^{-1}(g) = h(k_x, k_y) \]

**Gap function**

\[ \Delta_+ = \Delta_0\sin(k_y a)\sigma_0 \]

\[ \Delta_- = \Delta_0\sin(k_y a)\sigma_3 \]

\[ D_{k\pm}(g)\Delta_\pm(\vec{k})D_{-k\pm}^T(g) = \pm\Delta_\pm(\vec{k}), \]

Our model Hamiltonian can be viewed as a generalization of Kitaev model

\[
\Delta_+ = \Delta_0 \sin(k_y a) \sigma_0 \\
\Delta_- = \Delta_0 \sin(k_y a) \sigma_3
\]

Topological glide superconductors

- Energy spectrum for a slab configuration
In a 1D glide superconductors, there should be $4N$ bands for BdG Hamiltonian due to the glide symmetry and particle-hole symmetry.
### Topological classification

- Topological classification with and without time reversal symmetry

#### Without Time Reversal Symmetry (No TR)

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<th>NoGS(2D)</th>
<th>spinless</th>
<th>spin-$\frac{1}{2}$</th>
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#### With Time Reversal Symmetry (TR)

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Topological non-symmorphic insulators

- Topological insulating phase protected by non-symmorphic symmetry does not exist in 2D, but only in 3D.
- Two types of topological non-symmorphic insulators

\[ \text{pg: Fang and Fu (2015)} \]
\[ \text{pmg, pgg and p4g: Liu’s group (2014), (2015)} \]
Degeneracy and topological surface states

- Degeneracy can lead to non-trivial surface states. E.g. TI due to time reversal symmetry.
Topological non-symmorphic insulators

- Non-symmorphic symmetry can induce degeneracy
- An example in pmg group

Liu’s group (2014)
Two symmetry operations

- **z-direction mirror operation** $m_z: (x, y, z) \rightarrow (x, y, -z)$
- **x-direction glide operation** $g_x = \{\sigma_x | \vec{t}\}: (x, y, z) \rightarrow (-x, y, z + \frac{c}{2})$, $\vec{t} = (0,0,\frac{c}{2})$

**Anti-commutation relation**

$$g_x m_z = m_z g_x + t, \quad t = (0,0,c)$$

When $\vec{k} = (0, k_y, \frac{\pi}{c}) = (\frac{\pi}{a}, k_y, \frac{\pi}{c})$, $g_x m_z |\phi_k\rangle = e^{ik\cdot\vec{t}} m_z g_x |\phi_k\rangle$

$$\rightarrow \{m_z, g_x\} = 0 \quad \text{at } \bar{Z} \text{ and } \bar{U}$$
Degeneracy due to non-symmorphic symmetry

- Degeneracy due to the anti-commutation relation in a symmetry group

\[
[R, H] = 0, \quad [S, H] = 0, \quad \{R, S\} = 0
\]

If \( H|\phi\rangle = E|\phi\rangle \) and \( R|\phi\rangle = r|\phi\rangle \), \( S|\phi\rangle \) and \( |\phi\rangle \) are two orthogonal and degenerate eigen states.

\[
HS|\phi\rangle = SH|\phi\rangle = ES|\phi\rangle \quad \rightarrow \quad S|\phi\rangle \text{ is an eigen-state}
\]

\[
RS|\phi\rangle = -SR|\phi\rangle = -rS|\phi\rangle \quad \rightarrow \quad S|\phi\rangle \text{ is different from } |\phi\rangle.
\]

Mathematically, all high dimensional irreducible representations in a space symmetry group are due to non-commutation relations.
**Surface states in TNCIs**

- Non-symmorphic symmetry

Anti-commutation relation requires the degeneracy at $\overline{Z}$ and $\overline{U}$.

CXL, RXZ and BV (2014)

- Model Hamiltonian

$$H = H_A + H_B + H_{AB}$$

$$H_\eta = \sum_{(\vec{n}\vec{m})_{in,\alpha\beta}} t_{\vec{n}\vec{m}}^{\alpha\beta} c_{\alpha\eta}^{\dagger} c_{\beta\eta} + \sum_{\vec{n},\alpha} \epsilon_{\alpha} c_{\alpha\eta}^{\dagger} c_{\alpha\eta}$$

$$+ \sum_{\vec{n}} \delta_{\eta} M_1 \left( -i c_{\eta p_x \eta}^{\dagger} c_{\eta p_y \eta} + \text{H.c.} \right),$$

$$H_{AB} = \sum_{(\vec{n}\vec{m})_{AB,\alpha\beta}} \left( r_{\vec{n}\vec{m}}^{\alpha\beta} c_{\alpha\eta A}^{\dagger} c_{\beta\eta B} + \text{H.c.} \right),$$
Surface states in TNCIs

• Gapless surface states

Perform calculation of this Hamiltonian on a slab configuration.

CXL, RXZ and BV (2014)
Topological non-symmorphic insulators

- Bulk topological invariants can be defined by Wannier function centers. (Pfaffian does not work).

Flow of Wannier function centers

More topological non-symmorphic insulators

- Here we only focus on non-symmorphic symmetry, but there are more crystalline symmetries.
- There are 17 2D space groups and 230 3D space groups.
- Our classification of all possible topological crystalline insulators.

The classification of topological crystalline insulators based on representation theory

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\(^2\)Department of Physics, The Pennsylvania State University, University Park, Pennsylvania 16802-6300, USA;
# Classification of topological crystalline insulators

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<td>rh</td>
<td>$\Gamma$-$X$,$Y$: $C_{2v}$</td>
<td>$\Gamma$-$X$:$m_y$; $\Gamma$-$Y$:$m_x$</td>
<td>None</td>
</tr>
<tr>
<td>$pg$</td>
<td>re</td>
<td>$\Gamma$-$X$,$Y$-$M$: $g_y$</td>
<td>$\mathbb{Z}_2$</td>
<td>$\mathbb{Z}_2$</td>
</tr>
<tr>
<td>$p4m$</td>
<td>s</td>
<td>$\Gamma$,$M$: $C_{4v}$</td>
<td>$\Gamma$-$X$:$m_y$; $\Gamma$-$X$-$M$:$m_x$; $\Gamma$-$M$:$m_d$</td>
<td>$\mathbb{Z}_3$ ($\Gamma$, $M$ $\in$ $E$)</td>
</tr>
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<td></td>
<td></td>
<td></td>
<td>None ($\Gamma$ or $M$ $\notin$ $E$)</td>
</tr>
<tr>
<td>$p3lm$</td>
<td>h</td>
<td>$\Gamma$,$K$,$K'$:$C_{3v}$</td>
<td>$\Gamma$-$K$:$m_1$; $\Gamma$-$K'$:$m_2$; $K'$-$\Gamma$:$m_3$</td>
<td>$\mathbb{Z}_2$ ($3$ HSPs $\notin$ $E$)</td>
</tr>
<tr>
<td>$p6m$</td>
<td>h</td>
<td>$\Gamma$: $C_{6v}$</td>
<td>$\Gamma$-$K$:$m_1$; $\Gamma$-$M$:$m_2$; $K$-$M$:$m_3$</td>
<td>$\mathbb{Z}_2$ ($\Gamma \in E_i$ ($i=1,2$), $K \in E_i$, $M \in A_i(B_i)$)</td>
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<td></td>
<td>$\mathbb{Z}_2$ ($\Gamma \in E_i$ ($i=1,2$), $K \in E$)</td>
</tr>
<tr>
<td>$pgg$</td>
<td>re</td>
<td>$\Gamma$-$X$,$Y$-$M$: $g_y$</td>
<td>$\mathbb{Z}_2$ ($\Gamma \in E_i$ ($i=1,2$), $K \in E_i$)</td>
<td>$\mathbb{Z}_2$ ($\Gamma \in E_i$ ($i=1,2$), $K \in E_i$)</td>
</tr>
<tr>
<td>$pmg$</td>
<td>re</td>
<td>$\Gamma$-$X$,$Y$-$M$: $g_y$</td>
<td>$\mathbb{Z}_2$ ($\Gamma \in A_i$, $Y \in B_i$)</td>
<td>$\mathbb{Z}_2$ ($\Gamma \in A_i$, $Y \in B_i$)</td>
</tr>
<tr>
<td>$p4g$</td>
<td>s</td>
<td>$\Gamma$,$M$: $C_{4v}$</td>
<td>$\Gamma$-$X$:$g_y$; $X$-$M$:$g_z$; $\Gamma$-$M$:$g_d$</td>
<td>$\mathbb{Z}_3$ ($\Gamma$, $M$ $\in$ $E$)</td>
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<td>$\mathbb{Z}_2$ ($\Gamma$ $\in$ $E_i$, $\Gamma$ $\in A_i(B_i)$; $\Gamma$ $\rightarrow$ $M$)</td>
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<td>$\mathbb{Z}_2$ ($\Gamma$ $\rightarrow$ $M$)</td>
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<td>None ($\Gamma$, $M$ $\notin$ $E$)</td>
</tr>
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</table>
Conclusion and outlook

- We have shown the existence of new topological superconducting and insulating phases in non-symmorphic crystals.
- Possible material realization? Photonic crystals? Iron pnictide superconductors?


- More topological phases
  - Classification of topological crystalline superconductors
  - Classification of topological crystalline semi-metal phases (e.g. Dirac semi-metals)
  - Interacting topological phases and crystalline symmetry

Fu (2015)
Thanks for your attention!
**Topological states of matter**

- **Topological states for free fermions**
  - Topological states cannot be adiabatically connected to normal states without gap closing even though sharing the same symmetry.
  - Topological states have edge/surface modes at the boundary (surface or interface).
The gap function is classified by glide reflection

\[ D_{\vec{k}}(g) \Delta(\vec{k}) D_{-\vec{k}}^T(g) = \eta \Delta(\vec{k}), \quad \eta = \pm \]

\[ \Delta_{\alpha\beta}(k) = V_0 \langle \psi_{\alpha,-k} \psi_{\beta,k} \rangle \]

Glide symmetry for BdG Hamiltonian

\[ \mathcal{G}_\eta(\vec{k}) = \begin{pmatrix} D_{\vec{k}}(g) & 0 \\ 0 & \eta D_{-\vec{k}}^*(g) \end{pmatrix} \]

\[ \mathcal{G}_\eta(\vec{k}) H_{BdG} \mathcal{G}_\eta^+(\vec{k}) = H_{BdG} \]
### Glide parity

**Glide parities of particle-hole partners**

\[ H_{BdG} \psi = E \psi, \quad G(g) \psi = \delta_\eta e^{i \vec{k} \cdot \vec{\tau}} \psi, \quad \delta_\eta = \pm \]

\[ \tilde{\psi} = C \psi, \quad H_{BdG} \tilde{\psi} = -E \tilde{\psi}, \quad G(g) \tilde{\psi} = \eta \delta_* e^{-i \vec{k} \cdot \vec{\tau}} \tilde{\psi} \]

**Glide parities depend on the momentum**

For example, for \( \eta = + \), the glide parities for \( \psi \) and \( \tilde{\psi} \) are \( e^{i \vec{k} \cdot \vec{\tau}} \) and \( e^{-i \vec{k} \cdot \vec{\tau}} \), respectively. They can be the same \((\vec{k} \cdot \vec{\tau} = 0)\) or different \((\vec{k} \cdot \vec{\tau} = \frac{\pi}{2})\).
Topological superconductors

- Kitaev model for 1D p-wave superconductor

\[
H - \mu N = \sum_i t(\psi_i^+\psi_{i+1} + \psi_{i+1}^+\psi_i) - \mu \psi_i^+\psi_i + \Delta(\psi_i\psi_{i+1} + \psi_{i+1}^+\psi_i^+)
\]

\[
\psi_i = \gamma_{1i} + i\gamma_{2i}, \quad \gamma_{\alpha i}^+ = \gamma_{\alpha i}
\]

\[
t_1 = \mu, \quad t_2 = 2t = 2\Delta
\]

Kitaev (2001)

Su, Schrieffer and Heeger (1979)
**Topological superconductors**

- **Kitaev model for 1D p wave superconductor**

Kitaev (2001)

\[ H = 2i \sum_i t_1 \gamma_i \gamma_{2i} + t_2 \gamma_{2i} \gamma_{1i+1} \]

**Trivial phase**

\[ t_1 > t_2 \]

**Topological phase**

\[ t_1 < t_2 \]

Unpaired end Majorana zero modes
Topological mirror Superconductors

- Key step: mathematically, we can still define a new "mirror" symmetry operation for the BdG Hamiltonian.

\[ D(m)\Delta(k)D^T(m) = \eta\Delta(k), \quad \eta = \pm \]

\[ D(m) = \begin{pmatrix} D(m) & 0 \\ 0 & \eta D^*(m) \end{pmatrix}, \quad H_{BdG} = \begin{pmatrix} h(k) - \mu & \Delta(k) \\ \Delta^+(k) & -h^*(-k) + \mu \end{pmatrix} \]

\[ D(m)H_{BdG}D^+(m) = H_{BdG} \]
o Any eigen-state of the BdG Hamiltonian can have the definite mirror parity.

\[ \mathcal{D}(m)H_{BdG} \mathcal{D}^+(m) = H_{BdG} \]

\[ H_{BdG}\psi = E\psi, \quad \mathcal{D}(m)\psi = \delta_m\psi, \quad \delta_m = \pm \]

o The mirror parities of \( \psi \) and \( \tilde{\psi} \) are determined by \( \eta \).

\[ C\mathcal{D}(m)C^{-1} = \eta\mathcal{D}(m) \]

\[ \tilde{\psi} = C\psi, \quad H_{BdG}\tilde{\psi} = -E\tilde{\psi}, \quad \mathcal{D}(m)\tilde{\psi} = \eta\delta_m\tilde{\psi} \]
When $\eta = +$, $\psi$ and $\tilde{\psi}$ have the same mirror parity, while $\eta = -$, $\psi$ and $\tilde{\psi}$ have opposite mirror parities.
Topological non-symmorphic insulators

- Anti-commutation relation at $\bar{Z}$ and $\bar{U}$.

$$g_x m_z = m_z g_x + 2\vec{\tau}$$

When $\vec{k} = (0, k_y, \frac{\pi}{c}) = (\frac{\pi}{a}, k_y, \frac{\pi}{c})$,

$$g_x m_z |\phi_{\vec{k}}\rangle = e^{2i\vec{k} \cdot \vec{\tau}} m_z g_x |\phi_{\vec{k}}\rangle$$

$$= -m_z g_x |\phi_{\vec{k}}\rangle$$

$$\{m_z, g_x\} = 0 \text{ at } \bar{Z} \text{ and } \bar{U}$$

Liu’s group (2014)
Symmetry and topological states of matter

- Topological insulators and time reversal symmetry
  - Insulating bulk gap and helical edge states which are protected by time reversal symmetry.

Molenkamp's group, Science (2007)

Qi and Zhang, RMP (2011), Hasan and Kane, RMP (2010).